Homework 6

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Due Wed.March 9

1. Find some way to show nonrigorously that the correlation function of white noise is a delta function. Suggestion: approximate BM by a random walk with Gaussian increments of non-zero length, find the time series of the derivative of this walk, calculate the correlations and take a (formal) limit.

2. Consider the stochastic process $u = \xi \cos(t)$, where ξ is a random variable with mean 0 and variance 1. Find the mean and the covariance functions. Obviously this is not a stationary process. However, $\cos(t) = (e^{it} + e^{-it})/2$. How do you reconcile this with the construction we have of stationary processes as sums of exponentials?

3. Consider the differential equation $(u^2)_x = \epsilon u_{xx}$ on the real line, with the boundary conditions $u(-\infty) = u_0, u(+\infty) = -u_0$, where ϵ, u_0 are constants. Assume that u is a velocity, with units L/T, where L is a unit of length and T a unit of time. Find the units of ϵ. Because of the boundary conditions u does not have a usual Fourier transform, but one can define one by taking the Fourier transform of u' and dividing it by ik. Let $\hat{u}(k)$ be this Fourier transform of u. Define the energy spectrum by $E(k) = |\hat{u}(k)|^2$. Find the units of $E(k)$; show that the dimensionless quantity $E(k)k^2/u_0^2$ must be a function of the variable $k\epsilon/u_0$. Deduce that as you take the limit $\epsilon \to 0$, the spectrum converges to $E(k) \sim 1/k^2$.

4. Extend BM to $-\infty, +\infty$ by starting another BM from the origin backward in time. Consider the integral $\int_{\infty}^{+\infty} \psi(s-t)dw(s)$, where w is the extended BM and ψ some nice smooth function that decays to zero at ∞ fast. Check that the integral makes sense and has the same value in the Stratonovich and Ito calculi. It defines a stochastic process $u(t)$. Calculate its mean and covariance. Is it stationary in the wide sense? the narrow sense?