Rapid Topology Optimization using Reduced-Order Models

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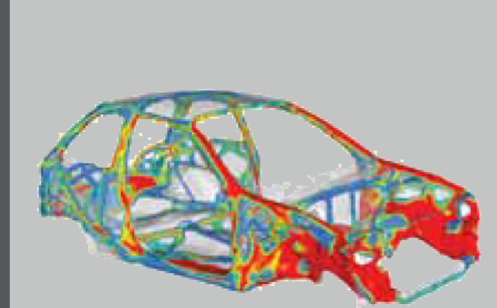
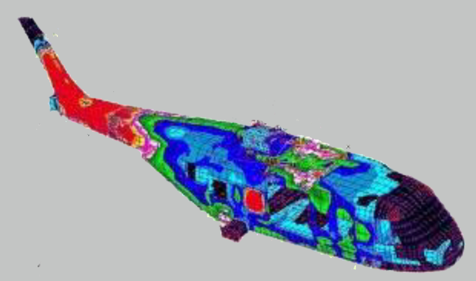
Motivation

- For industry-scale problems, topology optimization is a beneficial tool that is *time and resource intensive*
  - Large number of calls to structural solver usually required
  - Each structural call is expensive, especially for nonlinear 3D High-Dimensional Models (HDM)
- We introduce a Reduced-Order Model (ROM) as a surrogate for the structural model in a material topology optimization loop
  - Large speedups attained by leveraging cubic structure of the nonlinear equations (large deformations of specific materials)
  - ROM necessitates low-dimensional approx. to material distribution
    - Avoid online computations that scale with HDM
    - Small vector controlling material distribution, to be used as optimization variables





0-1 Material Topology Optimization

$$\underset{\chi \in \mathbb{R}^{n_{el}}}{\text{minimize}} \quad \mathcal{L}(\mathbf{u}(\chi), \chi)$$
$$\text{subject to } \mathbf{c}(\mathbf{u}(\chi), \chi) \leq 0$$

- $\mathbf{u}$ is implicitly defined as a function of χ through the HDM equation
$$\mathbf{f}^{int}(\mathbf{u}) = \mathbf{f}^{ext}$$
- $$\mathbb{C}^e = \mathbb{C}^e_0 \chi_e \qquad \rho^e = \rho^e_0 \chi_e \qquad \chi_e = \begin{cases} 0, & e \notin \Omega^* \\ 1, & e \in \Omega^* \end{cases}$$
- Assume geometric *nonlinearity* and linearity in the constitutive law

Reduced-Order Model

- Model Order Reduction (MOR) assumption
  - State vector lies in low-dimensional subspace defined by $\mathbf{\Phi}$
$$\mathbf{u} \approx \mathbf{\Phi}\mathbf{y}$$
- Galerkin projection
$$\mathbf{\Phi}^T \mathbf{f}^{int}(\mathbf{\Phi}\mathbf{y}) = \mathbf{\Phi}^T \mathbf{f}^{ext}$$

ROM Optimization Formulation

$$\underset{\alpha_r \in \mathbb{R}^{n_\alpha}}{\text{minimize}} \quad \mathcal{L}(\mathbf{y}(\alpha_r), \alpha_r)$$
$$\text{subject to } \mathbf{c}(\mathbf{y}(\alpha_r), \alpha_r) \leq 0$$

- $\mathbf{y}$ is implicitly defined as a function of α_r through the ROM equation
$$\mathbf{\Phi}^T \mathbf{f}^{int}(\mathbf{\Phi}\mathbf{y}) = \mathbf{\Phi}^T \mathbf{f}^{ext}$$

Internal Force - Cubic Polynomial in Displacements

The expression for the internal force is

$$\begin{aligned} \mathbf{f}^{int}_{jL} &= \int_{\Omega_0} \mathbf{P}_{ij} \frac{\partial \mathbf{N}_I}{\partial \mathbf{X}_i} \, d\mathbf{X} \\ &= \bar{\mathbf{A}}_{jtIL} \mathbf{u}_{tI} + \bar{\mathbf{B}}_{LI} \mathbf{u}_{jI} + \\ &\quad \bar{\mathbf{C}}_{LIJj} \mathbf{u}_{kI} \mathbf{u}_{kJ} + \hat{\mathbf{C}}_{ILQt} \mathbf{u}_{jQ} \mathbf{u}_{tI} + \bar{\mathbf{D}}_{IJQL} \mathbf{u}_{kI} \mathbf{u}_{kJ} \mathbf{u}_{jQ} \end{aligned}$$

where $\mathbf{N}_I(\mathbf{X})$ is the shape function corresponding to node I. Then

$$\bar{\mathbf{A}} = \bar{\mathbf{A}}(\Omega, \lambda(\mathbf{X})) \qquad \bar{\mathbf{B}} = \bar{\mathbf{B}}(\Omega, \mu(\mathbf{X}))$$
$$\bar{\mathbf{C}} = \bar{\mathbf{C}}(\Omega, \lambda(\mathbf{X}), \mu(\mathbf{X})) \qquad \hat{\mathbf{C}} = \hat{\mathbf{C}}(\Omega, \lambda(\mathbf{X}), \mu(\mathbf{X}))$$
$$\bar{\mathbf{D}} = \bar{\mathbf{D}}(\Omega, \lambda(\mathbf{X}), \mu(\mathbf{X}))$$

Material Distribution Representation

Let material distributions be represented with the basis functions:

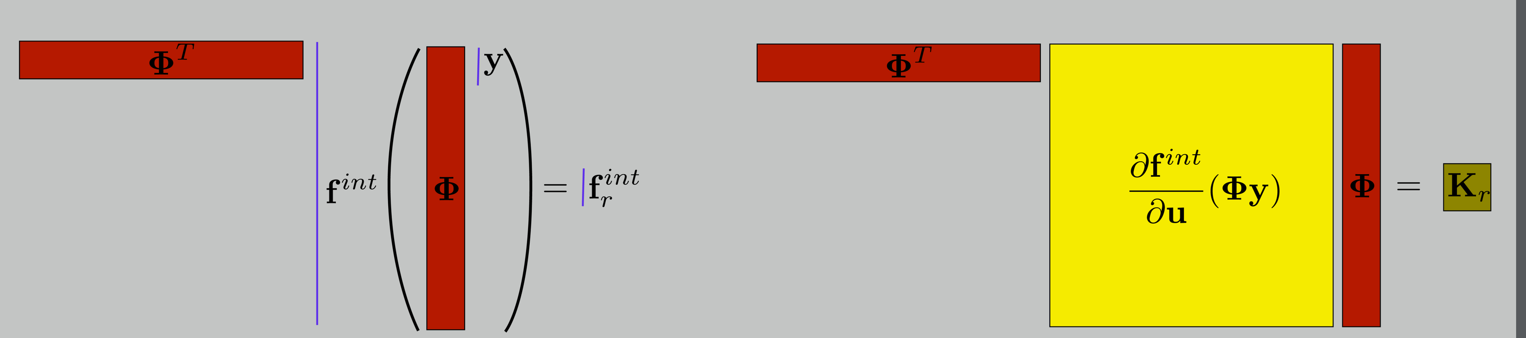
$$\begin{aligned} \lambda(\mathbf{X}) &= \phi^\lambda_i(\mathbf{X}) \alpha^r_i, & i &= 1, 2, \dots, n_\alpha \\ \mu(\mathbf{X}) &= \phi^\mu_i(\mathbf{X}) \alpha^r_i, & i &= 1, 2, \dots, n_\alpha \\ \rho(\mathbf{X}) &= \phi^\rho_i(\mathbf{X}) \alpha^r_i, & i &= 1, 2, \dots, n_\alpha \end{aligned}$$

Then

$$\bar{\mathbf{A}} = \bar{\mathbf{A}}(\Omega, \phi^\lambda_i) \alpha^r_i \qquad \bar{\mathbf{B}} = \bar{\mathbf{B}}(\Omega, \phi^\mu_i) \alpha^r_i$$
$$\bar{\mathbf{C}} = \bar{\mathbf{C}}(\Omega, \phi^\lambda_i, \phi^\mu_i) \alpha^r_i \qquad \hat{\mathbf{C}} = \hat{\mathbf{C}}(\Omega, \phi^\lambda_i, \phi^\mu_i) \alpha^r_i$$
$$\bar{\mathbf{D}} = \bar{\mathbf{D}}(\Omega, \phi^\lambda_i, \phi^\mu_i) \alpha^r_i$$

Reduced-Order Model via Precomputations

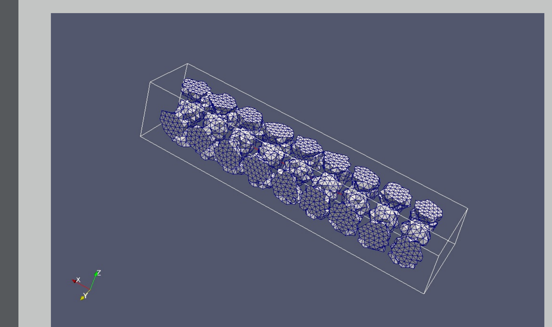
$$\mathbf{\Phi}^T \mathbf{f}^{int}(\mathbf{\Phi}\mathbf{y}) = \mathbf{\Phi}^T \mathbf{f}^{ext}$$



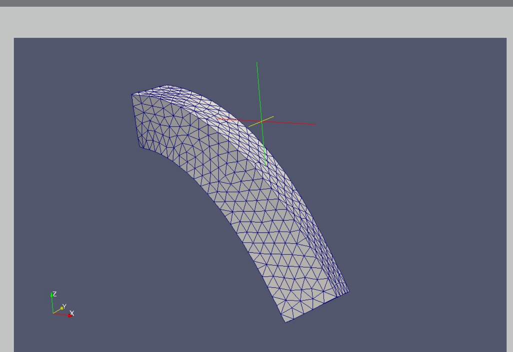
$$\left[\mathbf{\Phi}^T \mathbf{f}^{int}(\mathbf{\Phi}\mathbf{y})\right]_r = \beta_{rp}(\alpha_r) \mathbf{y}_p + \gamma_{rpq}(\alpha_r) \mathbf{y}_p \mathbf{y}_q + \omega_{rpqt}(\alpha_r) \mathbf{y}_p \mathbf{y}_q \mathbf{y}_t$$

- Amenable to material topology optimization
  - α^r provide control over material distribution
  - α^r optimization variables in 0-1 topology optimization
  - Vary material distribution only in the column space of $\mathbf{\Phi}^\lambda, \mathbf{\Phi}^\mu, \mathbf{\Phi}^\rho$
- Large speedups possible without hyperreduction, $\mathcal{O}(10^3)$
- *Currently* limited to StVK material, *Lagrangian* elements
- Cost/storage scales poorly with $k_{\mathbf{u}}$ (ROM size)
  - Offline cost scales as $\mathcal{O}(n_\alpha \cdot n_{el} \cdot k^4_{\mathbf{u}})$
  - Offline storage scales as $\mathcal{O}(n_\alpha \cdot k^4_{\mathbf{u}})$
  - Online storage scales as $\mathcal{O}(k^4_{\mathbf{u}})$

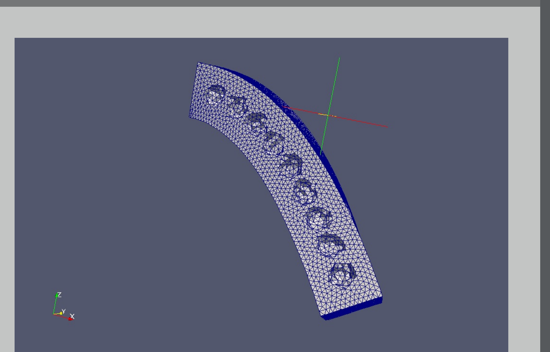
Cantilever - Weight Minimization



46 Material Snapshots



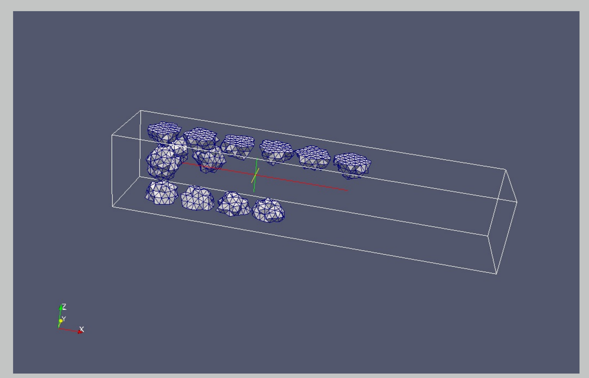
Deformed configuration (Initial Guess)



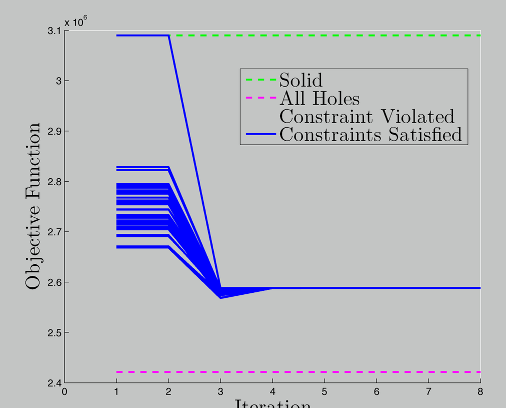
Deformed configuration (Optimal Solution)

- α^r determines existence of voids
- Vertical displacement bounds
- 38,664 dof
- Loads: bending, twisting, self-weight
- ROM size: $k_{\mathbf{u}} = 5$

	Online (sec)	Speedup
HDM	750	-
ROM	156	4.80
pROM	0.37	2,051

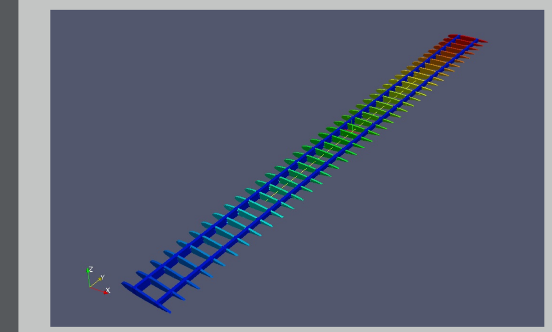


Remaining Material at Optimal Solution (Outer Matrix Removed)

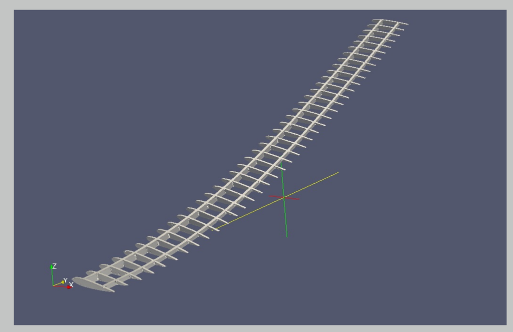


Convergence History - Multiple Initial Guesses

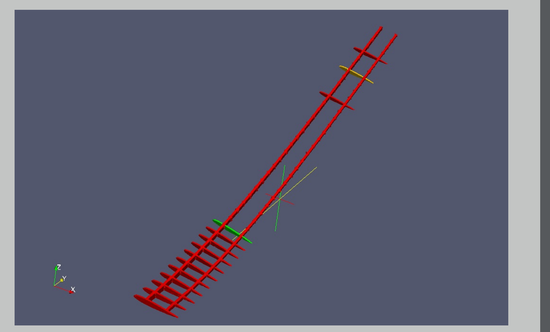
Wing Box Design - Weight Minimization



41 Material Snapshots



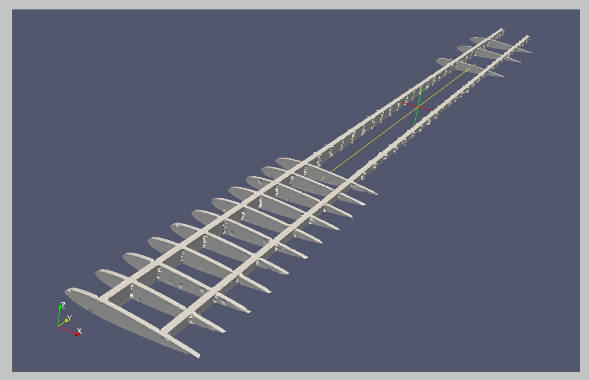
Deformed configuration (Initial Guess)



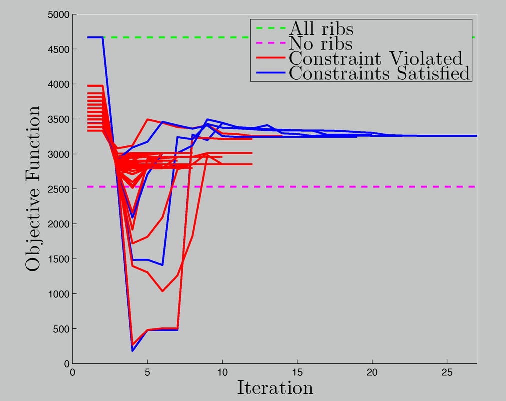
Deformed configuration (Optimal Solution)

- α^r determines placement of ribs
- Vertical/horizontal disp bounds
- Loads: bending (lift and drag loads), twisting, self-weight
- 86,493 dof
- ROM size: $k_{\mathbf{u}} = 5$

	Online (sec)	Speedup
HDM	811	-
ROM	376	2.16
pROM	1.51	538



Optimal Solution



Convergence History - Multiple Initial Guesses

Conclusion

- New method for material topology optimization using reduced-order models
  - $\mathcal{O}(10^3)$ speedup over HDM
- Potential to address large problems