Rapid Nonlinear Topology Optimization using Precomputed Reduced-Order Models

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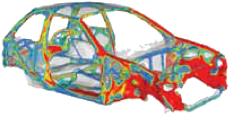
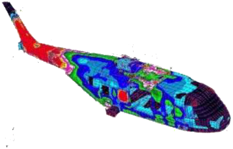
Motivation

- For industry-scale design problems, topology optimization is a beneficial tool that is *time and resource intensive*
  - Large number of calls to structural solver usually required
  - Each structural call is expensive, especially for nonlinear 3D High-Dimensional Models (HDM)
- Use a Reduced-Order Model (ROM) as a surrogate for the structural model in a material topology optimization loop
  - Large speedups over HDM realized





0-1 Material Topology Optimization

$$\begin{array}{ll} \underset{\boldsymbol{\chi} \in \mathbb{R}^{n_{el}}}{\text{minimize}} & \mathcal{L}(\mathbf{u}(\boldsymbol{\chi}), \boldsymbol{\chi}) \\ \text{subject to} & \mathbf{c}(\mathbf{u}(\boldsymbol{\chi}), \boldsymbol{\chi}) \leq 0 \end{array}$$

- $\mathbf{u}$ (structural displacements) is implicitly defined as a function of $\boldsymbol{\chi}$ through the HDM equation

$$\mathbf{f}^{int}(\mathbf{u}) = \mathbf{f}^{ext}$$

$$\mathbb{C}^e = \mathbb{C}_0^e \boldsymbol{\chi}_e \qquad \rho^e = \rho_0^e \boldsymbol{\chi}_e \qquad \boldsymbol{\chi}_e = \begin{cases} 0, & e \notin \Omega^* \\ 1, & e \in \Omega^* \end{cases}$$

- General nonlinear setting considered (geometric and material nonlinearities)

Reduced-Order Model

- Model Order Reduction (MOR) assumption
  - State vector lies in low-dimensional subspace defined by a Reduced-Order Basis (ROB) $\mathbf{\Phi} \in \mathbb{R}^{N \times k_{\mathbf{u}}}$

$$\mathbf{u} \approx \mathbf{\Phi}\mathbf{y}$$

  - $k_{\mathbf{u}} \ll N$
- N equations, $k_{\mathbf{u}}$ unknowns

$$\mathbf{f}^{int}(\mathbf{\Phi}\mathbf{y}) = \mathbf{f}^{ext}$$

- Galerkin projection

$$\mathbf{\Phi}^T\mathbf{f}^{int}(\mathbf{\Phi}\mathbf{y}) = \mathbf{\Phi}^T\mathbf{f}^{ext}$$

NL ROM Bottleneck - Internal Force

$$\boldsymbol{\Phi}^T \mathbf{f}^{int}(\boldsymbol{\Phi}\mathbf{y}) = \boldsymbol{\Phi}^T \mathbf{f}^{ext}$$

$$\boldsymbol{\Phi}^T \; \mathbf{f}^{int}\left(\boldsymbol{\Phi} \; \mathbf{y} \right) = \mathbf{f}_r^{int}$$

NL ROM Bottleneck - Tangent Stiffness

$$\boldsymbol{\Phi}^T \mathbf{f}^{int}(\boldsymbol{\Phi}\mathbf{y}) = \boldsymbol{\Phi}^T \mathbf{f}^{ext}$$

$$\boldsymbol{\Phi}^T \; \frac{\partial \mathbf{f}^{int}}{\partial \mathbf{u}}(\boldsymbol{\Phi}\mathbf{y}) \; \boldsymbol{\Phi} = \mathbf{K}_r$$

Approximation of reduced internal force, $\mathbf{\Phi}^T \mathbf{f}^{\text{int}}(\mathbf{\Phi y})$

- For general nonlinear problems, high-dimensional quantities cannot be precomputed since they change at every iteration
- For *polynomial* nonlinearities, there is an opportunity for precomputation
- Approach
  - Approximate $\mathbf{f}^r = \mathbf{\Phi}^T \mathbf{f}^{\text{int}}(\mathbf{\Phi y})$ by polynomial via Taylor series
    - We choose a third-order series
    - Exact representation of reduced internal force for *St. Venant-Kirchhoff* materials
  - Precompute coefficient tensors
  - Online operations will only involve *small* quantities
    - Remove online bottleneck
    - Pay price in offline phase

Taylor Series of $\mathbf{\Phi}^T \mathbf{f}^{\text{int}}(\mathbf{\Phi} \mathbf{y})$

Consider Taylor series expansion of $\mathbf{f}^r(\mathbf{y}) = \mathbf{\Phi}^T \mathbf{f}^{\text{int}}(\mathbf{\Phi}\mathbf{y})$ about $\bar{\mathbf{y}}$

$$\begin{aligned}
\mathbf{f}_i^r(\mathbf{y}) \approx \mathbf{f}_i^r(\bar{\mathbf{y}}) &+ \frac{\partial \mathbf{f}_i^r}{\partial \mathbf{y}_j}(\bar{\mathbf{y}}) \cdot (\mathbf{y} - \bar{\mathbf{y}})_j \\
&+ \frac{1}{2} \frac{\partial^2 \mathbf{f}_i^r}{\partial \mathbf{y}_j \partial \mathbf{y}_k}(\bar{\mathbf{y}}) \cdot (\mathbf{y} - \bar{\mathbf{y}})_j (\mathbf{y} - \bar{\mathbf{y}})_k \\
&+ \frac{1}{6} \frac{\partial^3 \mathbf{f}_i^r}{\partial \mathbf{y}_j \partial \mathbf{y}_k \partial \mathbf{y}_l}(\bar{\mathbf{y}}) \cdot (\mathbf{y} - \bar{\mathbf{y}})_j (\mathbf{y} - \bar{\mathbf{y}})_k (\mathbf{y} - \bar{\mathbf{y}})_l
\end{aligned}$$

Reduced Derivatives

- Reduced derivatives computable by:
  - Projection of *full* order derivatives
  - Directly via finite differences

$$
\begin{aligned}
\alpha_i = \mathbf{f}^r_i(\bar{\mathbf{y}}) &= \boldsymbol{\Phi}_{pi}\mathbf{f}^{\text{int}}_p(\boldsymbol{\Phi}\bar{\mathbf{y}}) \\
\beta_{ij} = \frac{\partial \mathbf{f}^r_i}{\partial \mathbf{y}_j}(\bar{\mathbf{y}}) &= \boldsymbol{\Phi}_{pi}\boldsymbol{\Phi}_{qj}\frac{\partial \mathbf{f}^{\text{int}}_p}{\partial \mathbf{u}_q}(\boldsymbol{\Phi}\bar{\mathbf{y}}) \\
\gamma_{ijk} = \frac{\partial^2 \mathbf{f}^r_i}{\partial \mathbf{y}_j \partial \mathbf{y}_k}(\bar{\mathbf{y}}) &= \boldsymbol{\Phi}_{pi}\boldsymbol{\Phi}_{qj}\boldsymbol{\Phi}_{rk}\frac{\partial \mathbf{f}^{\text{int}}_p}{\partial \mathbf{u}_q \partial \mathbf{u}_r}(\boldsymbol{\Phi}\bar{\mathbf{y}}) \\
\omega_{ijkl} = \frac{\partial^3 \mathbf{f}^r_i}{\partial \mathbf{y}_j \partial \mathbf{y}_k \partial \mathbf{y}_l}(\bar{\mathbf{y}}) &= \boldsymbol{\Phi}_{pi}\boldsymbol{\Phi}_{qj}\boldsymbol{\Phi}_{rk}\boldsymbol{\Phi}_{sl}\frac{\partial \mathbf{f}^{\text{int}}_p}{\partial \mathbf{u}_q \partial \mathbf{u}_r \partial \mathbf{u}_s}(\boldsymbol{\Phi}\bar{\mathbf{y}})
\end{aligned}
$$

Reduced internal force

Reduced internal force becomes

$$\begin{aligned}\mathbf{f}_i^r(\mathbf{y}) = \alpha_i &+ \beta_{ij}(\mathbf{y}-\bar{\mathbf{y}})_j \\ &+ \frac{1}{2}\gamma_{ijk}(\mathbf{y}-\bar{\mathbf{y}})_j(\mathbf{y}-\bar{\mathbf{y}})_k \\ &+ \frac{1}{6}\omega_{ijkl}(\mathbf{y}-\bar{\mathbf{y}})_j(\mathbf{y}-\bar{\mathbf{y}})_k(\mathbf{y}-\bar{\mathbf{y}})_l,\end{aligned}$$

which only depends on quantities scaling with the *reduced* dimension.

Reduced internal force - material dependence

- As written, the material properties for a given material are *baked into* the polynomial coefficients
- For notational simplicity, we consider two material parameters: ρ (density) and η

$$\begin{aligned}\boldsymbol{\alpha} &= \boldsymbol{\alpha}(\rho, \eta)\\ \boldsymbol{\beta} &= \boldsymbol{\beta}(\rho, \eta)\\ \boldsymbol{\gamma} &= \boldsymbol{\gamma}(\rho, \eta)\\ \boldsymbol{\omega} &= \boldsymbol{\omega}(\rho, \eta)\end{aligned}$$

- In the context of 0-1 topology optimization, $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\omega}$ need to be recomputed at each new distribution of ρ, η
  - Extremely expensive – destroy all speedup potential

Material Representation

- Recall the material parameters are *spatial distributions*, i.e. $\rho = \rho(\mathbf{X})$ and $\eta = \eta(\mathbf{X})$
- Define admissible distributions: $\{\boldsymbol{\phi}_i^{\rho}\}_{i=1}^{n}$, $\{\boldsymbol{\phi}_i^{\eta}\}_{i=1}^{n}$
    - Require

$$\rho(\mathbf{X}) = \boldsymbol{\phi}_i^{\rho}(\mathbf{X})\xi_i$$
$$\eta(\mathbf{X}) = \boldsymbol{\phi}_i^{\eta}(\mathbf{X})\xi_i$$

    - Many possible choices admissible distributions
        - Here, collected via *configuration snapshots*

Reduced internal force - material dependence

- Suppose the coefficient matrices depend *linearly* on material parameters
  - Can be accomplished by carefully choosing parameters (i.e. λ, μ instead of E, ν) or linearization via Taylor series
- Use material assumptions in reduced internal force

$$
\begin{aligned}
\mathbf{f}_i^r(\mathbf{y}) = & \sum_a \alpha_i\,(\boldsymbol{\phi}_a^\rho, \boldsymbol{\phi}_a^\eta)\xi_a \\
& + \sum_a \beta_{ij}\,(\boldsymbol{\phi}_a^\rho, \boldsymbol{\phi}_a^\eta)\xi_a(\mathbf{y} - \bar{\mathbf{y}})_j \\
& + \frac{1}{2}\sum_a \gamma_{ijk}\,(\boldsymbol{\phi}_a^\rho, \boldsymbol{\phi}_a^\eta)\xi_a(\mathbf{y} - \bar{\mathbf{y}})_j(\mathbf{y} - \bar{\mathbf{y}})_k \\
& + \frac{1}{6}\sum_a \omega_{ijkl}\,(\boldsymbol{\phi}_a^\rho, \boldsymbol{\phi}_a^\eta)\xi_a(\mathbf{y} - \bar{\mathbf{y}})_j(\mathbf{y} - \bar{\mathbf{y}})_k(\mathbf{y} - \bar{\mathbf{y}})_l
\end{aligned}
$$

- Quantities in blue can be precomputed offline

ROM Pre-computation Approach

$$\boxed{\mathbf{\Phi}^T \mathbf{f}^{int}(\mathbf{\Phi}\mathbf{y}) = \mathbf{\Phi}^T \mathbf{f}^{ext}}$$

Advantages

- Only need to solve small, cubic nonlinear system online
- Large speedups possible without hyperreduction, $\mathcal{O}(10^2)$
- Amenable to 0-1 material topology optimization

Disadvantages

- Offline cost scales as $\mathcal{O}(n_\alpha \cdot n_{el} \cdot k_{\mathbf{u}}^4)$
- Offline storage scales as $\mathcal{O}(n_\alpha \cdot k_{\mathbf{u}}^4)$
- Online storage scales as $\mathcal{O}(k_{\mathbf{u}}^4)$
- Can only vary material distribution in the subspace defined by the material snapshot vectors

Reduced Topology Optimization

$$\begin{aligned} \underset{\boldsymbol{\xi} \in \mathbb{R}^n}{\text{minimize}} \quad & \hat{\mathcal{L}}(\mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}) \\ \text{subject to} \quad & \hat{\mathbf{c}}(\mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}) \leq 0 \end{aligned}$$

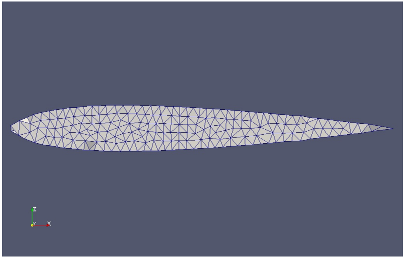
- $\mathbf{y}$ is implicitly defined as a function of $\boldsymbol{\xi}$ through the ROM equation

$$\boldsymbol{\Phi}^T \mathbf{f}^{int}(\boldsymbol{\Phi}\mathbf{y}) = \boldsymbol{\Phi}^T \mathbf{f}^{ext}$$

  which can be computed efficiently

Problem Setup

- Neo-Hookean material
- 90,799 tetrahedral elements
- 29,252 nodes, 86,493 dof
- Static simulation with load applied in 10 increments
- Loads: Bending (X- and Y- axis), Twisting, Self-Weight
- ROM size: $k_{\mathbf{u}} = 5$



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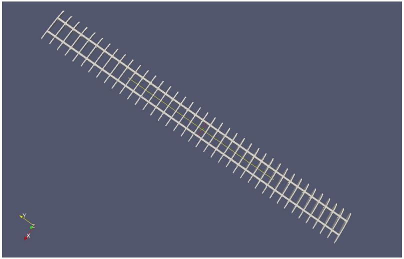

40 Ribs

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Simulation Results

- Single static simulation
- Training for ROMs: single static simulation (with load stepping) with *all* ribs
- Reproductive simulation

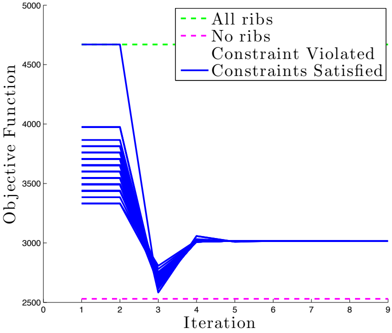
	Offline (s)	Online (s)	Speedup	Error (%)
HDM	-	674	-	-
ROM	0.988	412	1.64	0.002
ROM-precomp	6,724	1.19	566	5.54

Optimization Setup

- Minimize structural weight
- Constraint on maximum vertical horizontal displacements
- 41 Material Snapshots
  - 40 possible ribs
  - two spars jointly

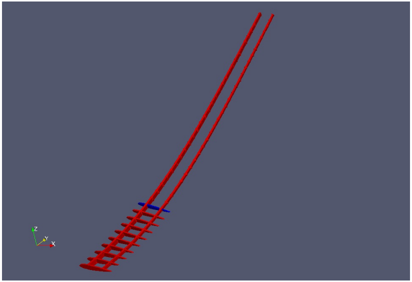
Material Snapshots

Optimization Results



Optimization Iterates

Optimization Results



Deformed Configuration (Optimal Solution)

	Initial Guess	Optimal Solution
Structural Weight	4.67×10^3	3.02×10^3
Constraint Violation	0	7.10×10^{-23}

Conclusion and Future Work

- New method for material topology optimization using reduced-order models
  - Applicable in nonlinear setting
  - $\mathcal{O}(10^2)$ speedup over HDM
- Strongly enforce manufacturability constraints
  - selection of material snapshots
- Address large problems
- Investigate extending method to more sophisticated topology optimization techniques