Accelerating PDE-Constrained Optimization Problems using Adaptive Reduced-Order Models

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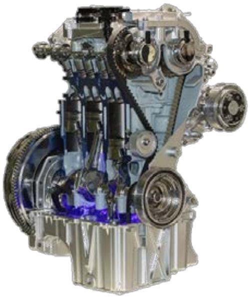
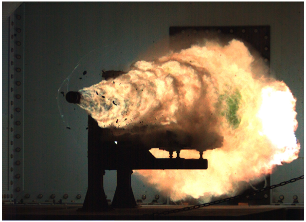
Multiphysics Optimization Key Player in Next-Gen Problems

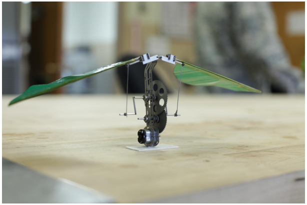
Current interest in computational physics reaches far beyond analysis of a single configuration of a physical system into ***design*** *(shape and topology[1]),* ***control****, and* ***uncertainty quantification***



Engine System



EM Launcher

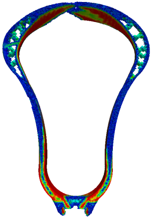


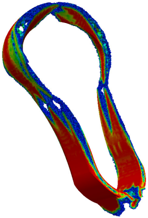
Micro-Aerial Vehicle

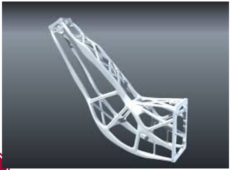
[1] Emergence of additive manufacturing technologies has made topology optimization increasingly relevant, particularly in DOE.

Topology Optimization and Additive Manufacturing[2]

- Emergence of AM has made TO an increasingly relevant topic
- AM+TO lead to highly efficient designs that could not be realized previously
- Challenges: smooth topologies require **very fine** meshes and modeling of complex **manufacturing process**







[2] *MIT Technology Review*, Top 10 Technological Breakthrough 2013

PDE-Constrained Optimization I

Goal: Rapidly solve PDE-constrained optimization problem of the form

$$\begin{aligned} \underset{\boldsymbol{u} \in \mathbb{R}^{n_{\boldsymbol{u}}},\ \boldsymbol{\mu} \in \mathbb{R}^{n_{\boldsymbol{\mu}}}}{\text{minimize}} \quad & \mathcal{J}(\boldsymbol{u},\ \boldsymbol{\mu}) \\ \text{subject to} \quad & \boldsymbol{r}(\boldsymbol{u},\ \boldsymbol{\mu}) = 0 \end{aligned}$$

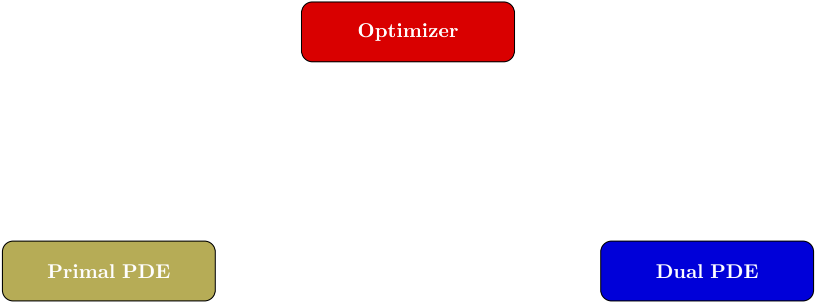
where

- $\boldsymbol{r} : \mathbb{R}^{n_{\boldsymbol{u}}} \times \mathbb{R}^{n_{\boldsymbol{\mu}}} \rightarrow \mathbb{R}^{n_{\boldsymbol{u}}}$ is the discretized partial differential equation
- $\mathcal{J} : \mathbb{R}^{n_{\boldsymbol{u}}} \times \mathbb{R}^{n_{\boldsymbol{\mu}}} \rightarrow \mathbb{R}$ is the objective function
- $\boldsymbol{u} \in \mathbb{R}^{n_{\boldsymbol{u}}}$ is the PDE state vector
- $\boldsymbol{\mu} \in \mathbb{R}^{n_{\boldsymbol{\mu}}}$ is the vector of parameters

red indicates a large-scale quantity, $\mathcal{O}(mesh)$

Nested Approach to PDE-Constrained Optimization

Virtually all expense emanates from primal/dual PDE solvers



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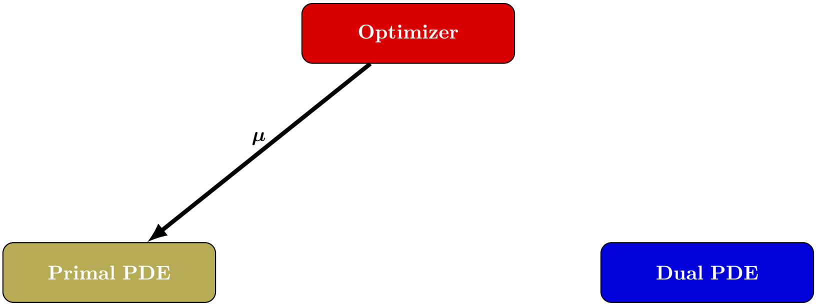

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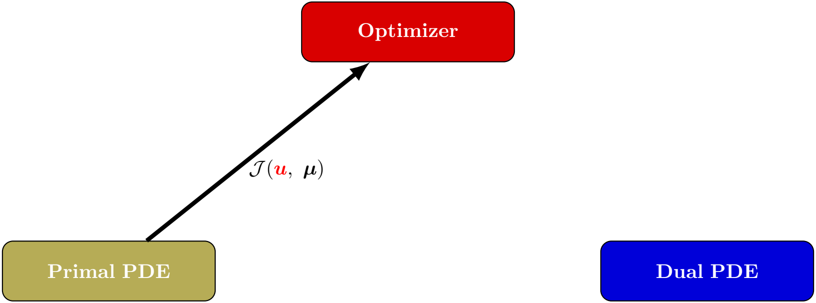

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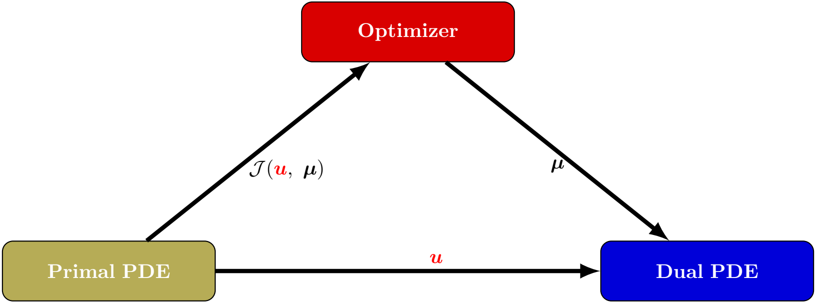

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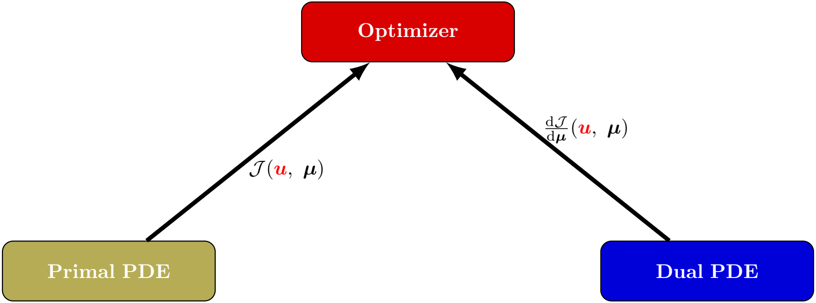


Projection-Based Model Reduction to Reduce PDE Size

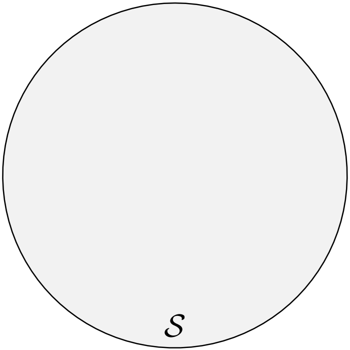
- Model Order Reduction (MOR) assumption: *state vector lies in low-dimensional subspace*

$$\boldsymbol{u} \approx \boldsymbol{\Phi_u} \boldsymbol{u}_r \qquad \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\mu}} \approx \boldsymbol{\Phi_u} \frac{\partial \boldsymbol{u}_r}{\partial \boldsymbol{\mu}}$$

where
  - $\boldsymbol{\Phi_u} = \begin{bmatrix} \boldsymbol{\phi}_{\boldsymbol{u}}^1 & \cdots & \boldsymbol{\phi}_{\boldsymbol{u}}^{k_{\boldsymbol{u}}} \end{bmatrix} \in \mathbb{R}^{n_{\boldsymbol{u}} \times k_{\boldsymbol{u}}}$ is the reduced basis
  - $\boldsymbol{u}_r \in \mathbb{R}^{k_{\boldsymbol{u}}}$ are the reduced coordinates of $\boldsymbol{u}$
  - $n_{\boldsymbol{u}} \gg k_{\boldsymbol{u}}$
- Substitute assumption into High-Dimensional Model (HDM), $\boldsymbol{r}(\boldsymbol{u},\ \boldsymbol{\mu}) = 0$, and project onto test subspace $\boldsymbol{\Psi_u} \in \mathbb{R}^{n_{\boldsymbol{u}} \times k_{\boldsymbol{u}}}$

$$\boldsymbol{\Psi_u}^T \boldsymbol{r}(\boldsymbol{\Phi_u} \boldsymbol{u}_r,\ \boldsymbol{\mu}) = 0$$

Connection to Finite Element Method: Hierarchical Subspaces



- $\mathcal{S}$ - infinite-dimensional trial space

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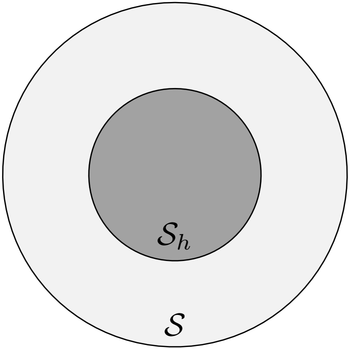

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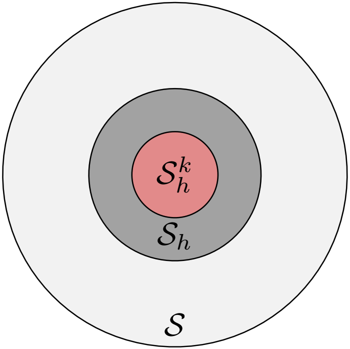
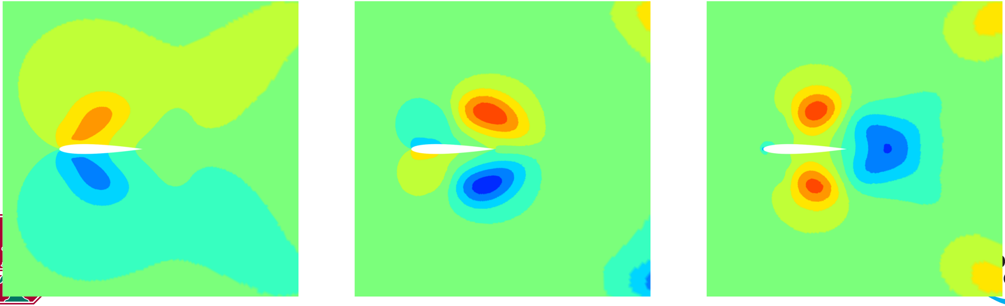


- $\mathcal{S}$ - infinite-dimensional trial space
- $\mathcal{S}_h$ - (large) finite-dimensional trial space
- $\mathcal{S}_h^k$ - (small) finite-dimensional trial space
- $\mathcal{S}_h^k \subset \mathcal{S}_h \subset \mathcal{S}$

Few Global, Data-Driven Basis Functions v. Many Local Ones

- Instead of using traditional *local* shape functions (e.g., FEM), use *global* shape functions
- Instead of a-priori, analytical shape functions, leverage data-rich computing environment by using *data-driven* modes



Definition of $\boldsymbol{\Phi}_{\boldsymbol{u}}$: Data-Driven Reduction

State-Sensitivity Proper Orthogonal Decomposition (POD)

- Collect state and sensitivity snapshots by sampling HDM

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{u}(\boldsymbol{\mu}_1) & \boldsymbol{u}(\boldsymbol{\mu}_2) & \cdots & \boldsymbol{u}(\boldsymbol{\mu}_n) \end{bmatrix}$$
$$\boldsymbol{Y} = \begin{bmatrix} \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\mu}}(\boldsymbol{\mu}_1) & \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\mu}}(\boldsymbol{\mu}_2) & \cdots & \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\mu}}(\boldsymbol{\mu}_n) \end{bmatrix}$$

- Use Proper Orthogonal Decomposition to generate reduced basis for each individually

$$\boldsymbol{\Phi}_{\boldsymbol{X}} = \mathrm{POD}(\boldsymbol{X})$$
$$\boldsymbol{\Phi}_{\boldsymbol{Y}} = \mathrm{POD}(\boldsymbol{Y})$$

- Concatenate and orthogonalize to get reduced-order basis

$$\boldsymbol{\Phi}_{\boldsymbol{u}} = \mathrm{QR}\left(\begin{bmatrix} \boldsymbol{u}(\boldsymbol{\mu}^*) & \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\mu}}(\boldsymbol{\mu}^*) & \boldsymbol{\Phi}_{\boldsymbol{X}} & \boldsymbol{\Phi}_{\boldsymbol{Y}} \end{bmatrix}\right)$$

Definition of $\boldsymbol{\Psi}_{\boldsymbol{u}}$: Minimum-Residual ROM

Least-Squares Petrov-Galerkin (LSPG)[3] projection

$$\boldsymbol{\Psi}_{\boldsymbol{u}} = \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{u}} \boldsymbol{\Phi}_{\boldsymbol{u}}$$

Minimum-Residual Property

A ROM possesses the minimum-residual property if $\boldsymbol{\Psi}_{\boldsymbol{u}} \boldsymbol{r}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r,\ \boldsymbol{\mu}) = 0$ is equivalent to the optimality condition of ($\Theta \succ 0$)

$$\underset{\boldsymbol{u}_r \in \mathbb{R}^{k_{\boldsymbol{u}}}}{\text{minimize}} \quad ||\boldsymbol{r}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r,\ \boldsymbol{\mu})||_{\Theta}$$

- Implications
  - Recover exact solution when basis not truncated (consistent[3])
  - Monotonic improvement of solution as basis size increases
  - Ensures sensitivity information in $\boldsymbol{\Phi}_{\boldsymbol{u}}$ cannot degrade state approximation[4]
- LSPG possesses minimum-residual property

[3] [Bui-Thanh et al., 2008]
[4] [Fahl, 2001]

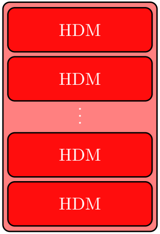
Offline-Online Approach to Optimization

Schematic

$\boldsymbol{\mu}$-space

Breakdown of Computational Effort

Offline-Online Approach to Optimization



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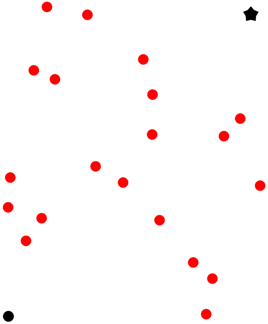

μ-space

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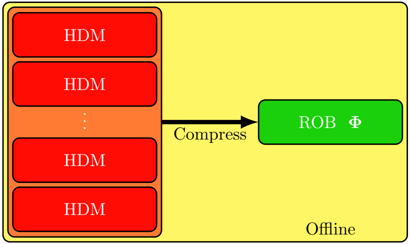

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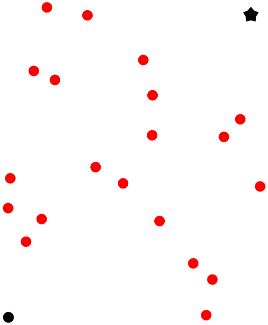

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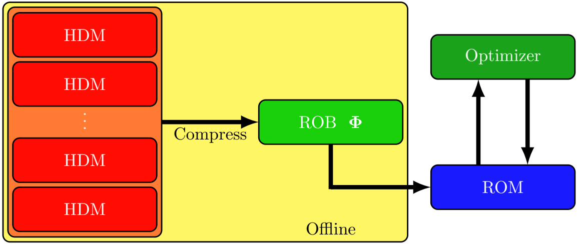

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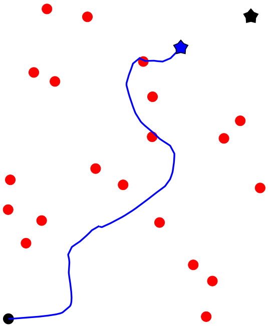
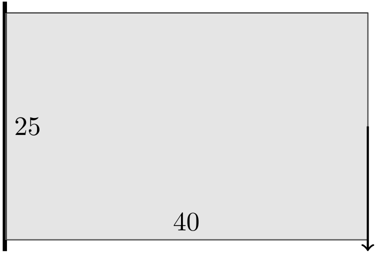


μ-space

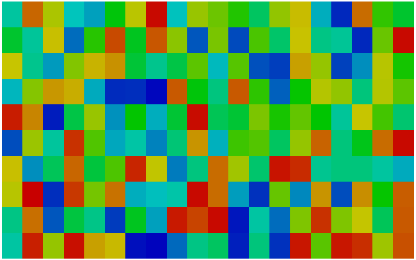
Breakdown of Computational Effort

Numerical Demonstration: Offline-Online Breakdown

- Parameter reduction ($\boldsymbol{\Phi}_{\boldsymbol{\mu}}$)
  - *apriori spatial clustering*
  - $k_{\boldsymbol{\mu}} = 200$
- *Greedy* Training
  - 5000 candidate points (LHS)
  - 50 snapshots
  - Error indicator: $||\mathbf{r}(\boldsymbol{\Phi}_{\boldsymbol{u}}\boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}}\boldsymbol{\mu}_r||$
- State reduction ($\boldsymbol{\Phi}_{\boldsymbol{u}}$)
  - POD
  - $k_{\boldsymbol{u}} = 25$
  - Polynomialization acceleration

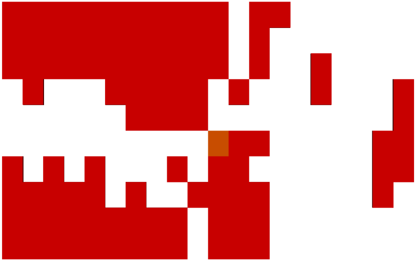


Stiffness maximization, volume constraint

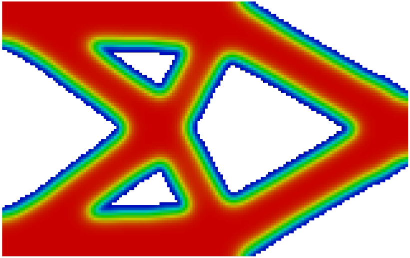


Parametrization with $k_{\boldsymbol{\mu}} = 200$

Numerical Demonstration: Offline-Online Breakdown



Optimal Solution (ROM)



Optimal Solution (HDM)

HDM Solution	**ROB** Construction	**Greedy** Algorithm	**ROM** Optimization
2.84×10^3 s	5.48×10^4 s	1.67×10^5 s	30 s
1.26%	24.36%	74.37%	0.01%

HDM Optimization: 1.97×10^4 s

ROM-Based Trust-Region Framework for Optimization

Schematic

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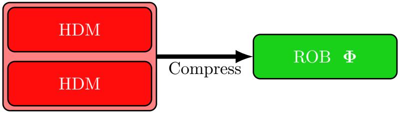

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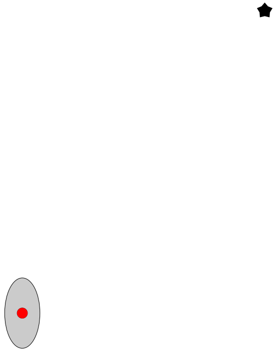

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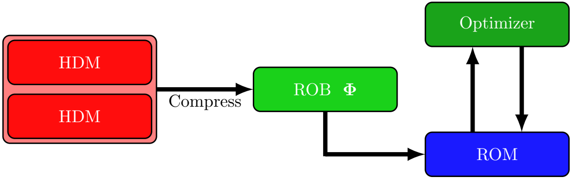

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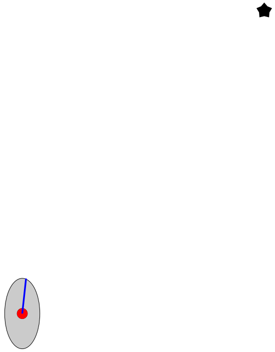

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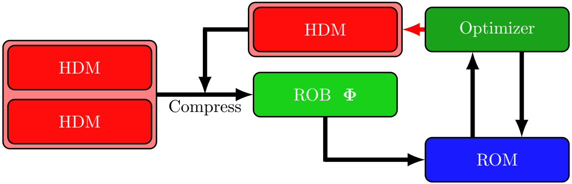

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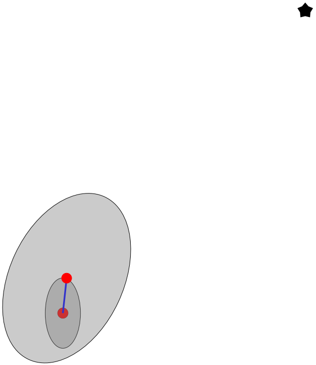

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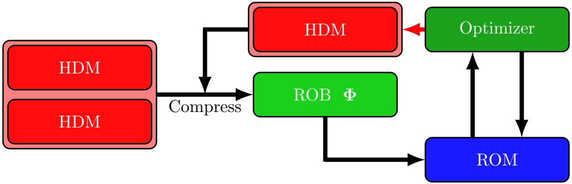

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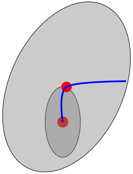

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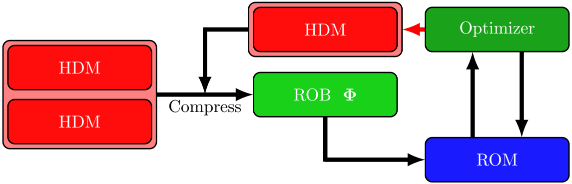

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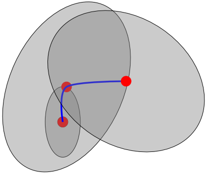

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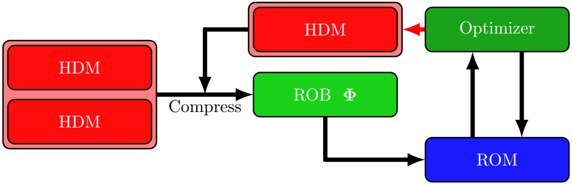

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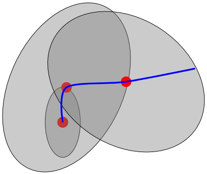

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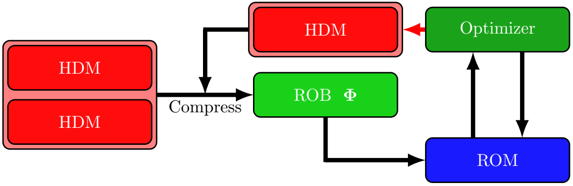

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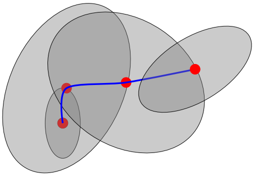

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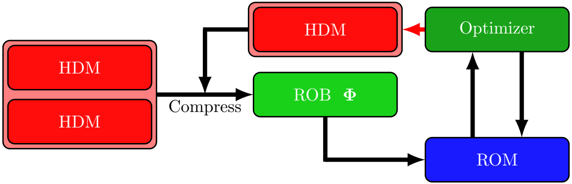

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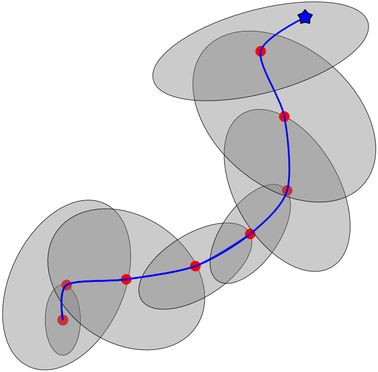

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Breakdown of Computational Effort

Non-Quadratic Trust-Region Method with Adaptive Reduced-Order Models

1: **Initialization**: Build $\boldsymbol{\Phi}_{\boldsymbol{u}}$ from *sparse* training

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2: **Step computation**: Approximately solve the reduced optimization problem with non-quadratic trust-region for a candidate, $\hat{\boldsymbol{\mu}}_k$

$$\underset{\boldsymbol{u}_r \in \mathbb{R}^{k_{\boldsymbol{u}}},\ \boldsymbol{\mu} \in \mathbb{R}^{n_{\boldsymbol{\mu}}}}{\text{minimize}} \mathcal{J}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r,\ \boldsymbol{\mu}) \qquad \text{subject to} \quad \boldsymbol{\Psi}_{\boldsymbol{u}}^T \boldsymbol{r}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r,\ \boldsymbol{\mu}) = 0$$

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$$||\boldsymbol{r}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r,\ \boldsymbol{\mu})|| \leq \Delta_k$$

3: **Step acceptance**: Compute

$$\rho_k = \frac{\mathcal{J}(\boldsymbol{u}(\boldsymbol{\mu}_k), \boldsymbol{\mu}_k) - \mathcal{J}(\boldsymbol{u}(\hat{\boldsymbol{\mu}}_k), \hat{\boldsymbol{\mu}}_k)}{\mathcal{J}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r(\boldsymbol{\mu}_k), \boldsymbol{\mu}_k) - \mathcal{J}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r(\hat{\boldsymbol{\mu}}_k), \hat{\boldsymbol{\mu}}_k)}$$

if $\rho_k \geq \eta_0$ **then** $\boldsymbol{\mu}_{k+1} = \hat{\boldsymbol{\mu}}_k$ **else** $\boldsymbol{\mu}_{k+1} = \boldsymbol{\mu}_k$ **end if**

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if $\rho_k \geq \eta_0$ **then** $\boldsymbol{\mu}_{k+1} = \hat{\boldsymbol{\mu}}_k$ **else** $\boldsymbol{\mu}_{k+1} = \boldsymbol{\mu}_k$ **end if**

4: **Trust-region update**:

if $\rho_k \leq \eta_1$ **then** $\Delta_{k+1} \in (0, \gamma||\boldsymbol{r}(\mathbf{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r(\hat{\boldsymbol{\mu}}_k), \hat{\boldsymbol{\mu}}_k)||]$ **end if**

if $\rho_k \in (\eta_1, \eta_2)$ **then** $\Delta_{k+1} \in [\gamma||\boldsymbol{r}(\mathbf{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r(\hat{\boldsymbol{\mu}}_k), \hat{\boldsymbol{\mu}}_k)||, \Delta_k]$ **end if**

if $\rho_k \geq \eta_2$ **then** $\Delta_{k+1} \in [\Delta_k, \Delta_{\max}]$ **end if**

Non-Quadratic Trust-Region Method with Adaptive Reduced-Order Models

1: **Initialization**: Build $\boldsymbol{\Phi}_{\boldsymbol{u}}$ from *sparse* training

2: **Step computation**: Approximately solve the reduced optimization problem with non-quadratic trust-region for a candidate, $\hat{\boldsymbol{\mu}}_k$

$$\underset{\boldsymbol{u}_r \in \mathbb{R}^{k_{\boldsymbol{u}}},\ \boldsymbol{\mu} \in \mathbb{R}^{n_{\boldsymbol{\mu}}}}{\text{minimize}} \ \mathcal{J}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r,\ \boldsymbol{\mu}) \qquad \text{subject to} \quad \begin{aligned} &\boldsymbol{\Psi}_{\boldsymbol{u}}^T \boldsymbol{r}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r,\ \boldsymbol{\mu}) = 0 \\ &||\boldsymbol{r}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r,\ \boldsymbol{\mu})|| \leq \Delta_k \end{aligned}$$

3: **Step acceptance**: Compute

$$\rho_k = \frac{\mathcal{J}(\boldsymbol{u}(\boldsymbol{\mu}_k), \boldsymbol{\mu}_k) - \mathcal{J}(\boldsymbol{u}(\hat{\boldsymbol{\mu}}_k), \hat{\boldsymbol{\mu}}_k)}{\mathcal{J}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r(\boldsymbol{\mu}_k), \boldsymbol{\mu}_k) - \mathcal{J}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r(\hat{\boldsymbol{\mu}}_k), \hat{\boldsymbol{\mu}}_k)}$$

if $\rho_k \geq \eta_0$ **then** $\boldsymbol{\mu}_{k+1} = \hat{\boldsymbol{\mu}}_k$ **else** $\boldsymbol{\mu}_{k+1} = \boldsymbol{\mu}_k$ **end if**

4: **Trust-region update**:

if $\rho_k \leq \eta_1$ **then** $\Delta_{k+1} \in (0, \gamma||\boldsymbol{r}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r(\hat{\boldsymbol{\mu}}_k), \hat{\boldsymbol{\mu}}_k)||]$ **end if**

if $\rho_k \in (\eta_1, \eta_2)$ **then** $\Delta_{k+1} \in [\gamma||\boldsymbol{r}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r(\hat{\boldsymbol{\mu}}_k), \hat{\boldsymbol{\mu}}_k)||, \Delta_k]$ **end if**

if $\rho_k \geq \eta_2$ **then** $\Delta_{k+1} \in [\Delta_k, \Delta_{\max}]$ **end if**

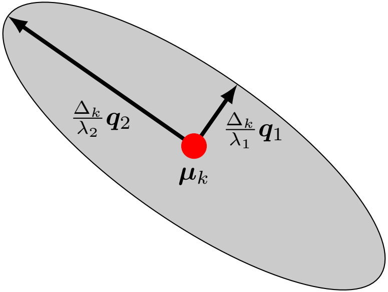
5: **Model update**: Enrich $\boldsymbol{\Phi}_{\boldsymbol{u}}$ with $\boldsymbol{u}(\hat{\boldsymbol{\mu}}_k)$ and $\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\mu}}(\hat{\boldsymbol{\mu}}_k)$

Residual-Based Trust-Region Interpretation

Let $\hat{\boldsymbol{r}}(\boldsymbol{\mu}) = \boldsymbol{r}(\boldsymbol{\Phi}_{\boldsymbol{u}}\boldsymbol{u}_r(\boldsymbol{\mu}), \boldsymbol{\mu})$ and $\boldsymbol{A}_k = \frac{\partial \hat{\boldsymbol{r}}}{\partial \boldsymbol{\mu}}(\boldsymbol{\mu}_k)^T \frac{\partial \hat{\boldsymbol{r}}}{\partial \boldsymbol{\mu}}(\boldsymbol{\mu}_k) = \boldsymbol{Q}_k \boldsymbol{\Lambda}_k^2 \boldsymbol{Q}_k^T$.

Then, to first order[5],

$$||\hat{\boldsymbol{r}}(\boldsymbol{\mu})||_2 = ||\frac{\partial \hat{\boldsymbol{r}}}{\partial \boldsymbol{\mu}}(\boldsymbol{\mu}_k)(\boldsymbol{\mu} - \boldsymbol{\mu}_k)||_2 = ||\boldsymbol{\mu} - \boldsymbol{\mu}_k||_{\boldsymbol{A}_k} \leq \Delta_k$$



Annotated schematic of trust-region: $\boldsymbol{q}_i = \boldsymbol{Q}_k \boldsymbol{e}_i$ and $\lambda_i = \boldsymbol{e}_i^T \boldsymbol{\Lambda}_k \boldsymbol{e}_i$

[5] assuming $\hat{\boldsymbol{r}}(\boldsymbol{\mu}_k) = 0$, i.e., ROM exact at trust-region center

Convergence to Critical Point of *Unreduced* Problem

Lim-Inf Convergence to Critical Point of Unreduced Optimization Problem

Let $\{\boldsymbol{\mu}_k\}$ be a sequence of iterations produced by the Algorithm and suppose

- $\mathcal{J}(\boldsymbol{u}(\boldsymbol{\mu}_k), \boldsymbol{\mu}_k) = \mathcal{J}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r(\boldsymbol{\mu}_k), \boldsymbol{\mu}_k)$
- There exists $\xi > 0$ such that

$$||\nabla \mathcal{J}(\boldsymbol{u}(\boldsymbol{\mu}_k), \boldsymbol{\mu}_k) - \nabla \mathcal{J}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r(\boldsymbol{\mu}_k), \boldsymbol{\mu}_k)|| \leq \xi ||\nabla \mathcal{J}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r(\boldsymbol{\mu}_k), \boldsymbol{\mu}_k)||$$

- There exists $\zeta > 0$ such that for all $\boldsymbol{\mu} \in \{\boldsymbol{\mu} \mid ||\boldsymbol{r}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r(\boldsymbol{\mu}), \boldsymbol{\mu})|| \leq \Delta_k\}$

$$|\mathcal{J}(\boldsymbol{u}(\boldsymbol{\mu}), \boldsymbol{\mu}) - \mathcal{J}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r(\boldsymbol{\mu}), \boldsymbol{\mu})| \leq \zeta ||\boldsymbol{r}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r(\boldsymbol{\mu}), \boldsymbol{\mu})||.$$

Then

$$\liminf_{k \to \infty} ||\nabla \mathcal{J}(\boldsymbol{u}(\boldsymbol{\mu}_k), \ \boldsymbol{\mu}_k)|| = 0$$

Assumptions of Convergence Theory Hold

If $\boldsymbol{\mu}_k$ is a *training* point, then

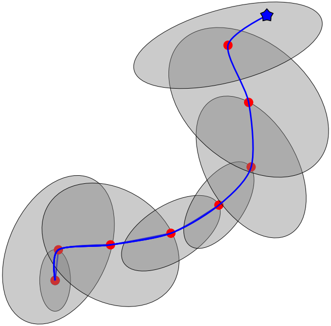
- **Minimum-residual** formulation for the **primal** reduced-order model implies

$$\mathcal{J}(\boldsymbol{u}(\boldsymbol{\mu}_k), \boldsymbol{\mu}_k) = \mathcal{J}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r(\boldsymbol{\mu}_k), \boldsymbol{\mu}_k)$$

- **Minimum-residual** formulation for the reduced-order model **sensitivity** implies

$$\nabla \mathcal{J}(\boldsymbol{u}(\boldsymbol{\mu}_k), \boldsymbol{\mu}_k) = \nabla \mathcal{J}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r(\boldsymbol{\mu}_k), \boldsymbol{\mu}_k)$$

- Standard **residual-based error estimation** implies, for some $\zeta > 0$,

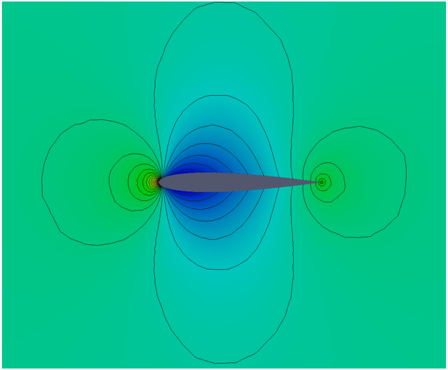


$\boldsymbol{\mu}$-space

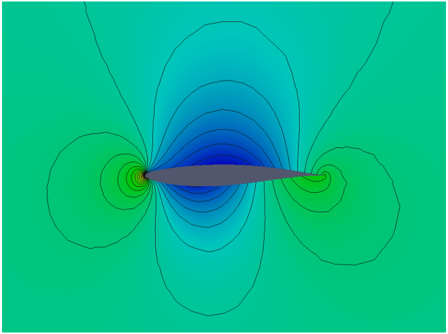
$$|\mathcal{J}(\boldsymbol{u}(\boldsymbol{\mu}), \boldsymbol{\mu}) - \mathcal{J}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r(\boldsymbol{\mu}), \boldsymbol{\mu})| \leq \zeta ||\boldsymbol{r}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r(\boldsymbol{\mu}), \boldsymbol{\mu})||$$

Compressible, Inviscid Airfoil Inverse Design

Pressure discrepancy minimization (Euler equations)



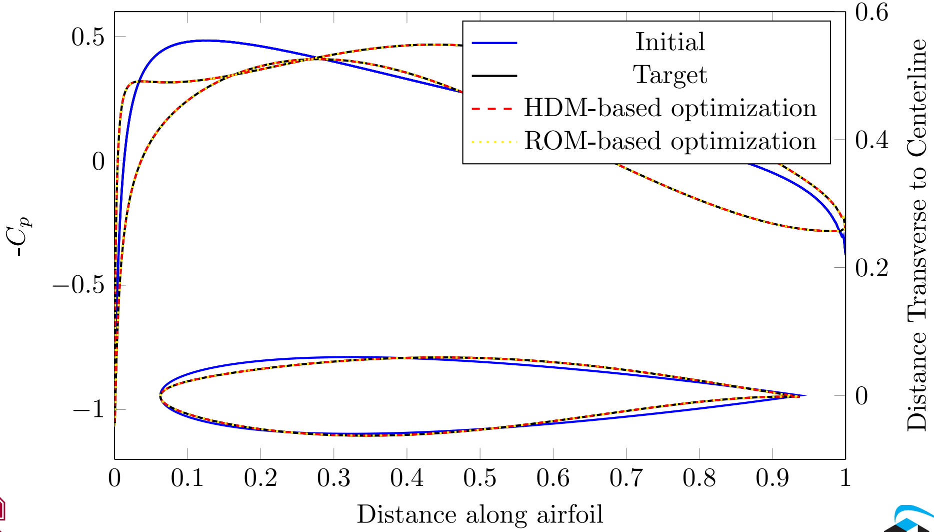
NACA0012: Initial



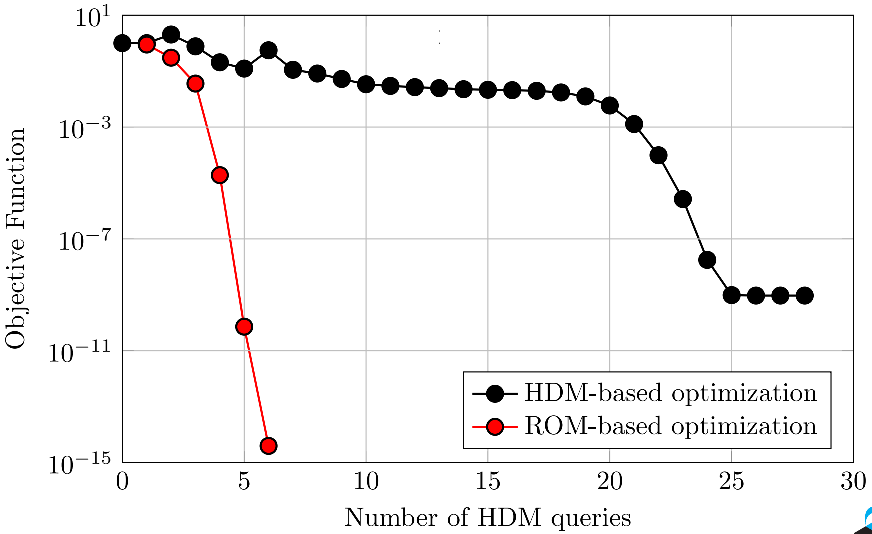
RAE2822: Target

Pressure field for airfoil configurations at $M_\infty = 0.5$, $\alpha = 0.0^\circ$

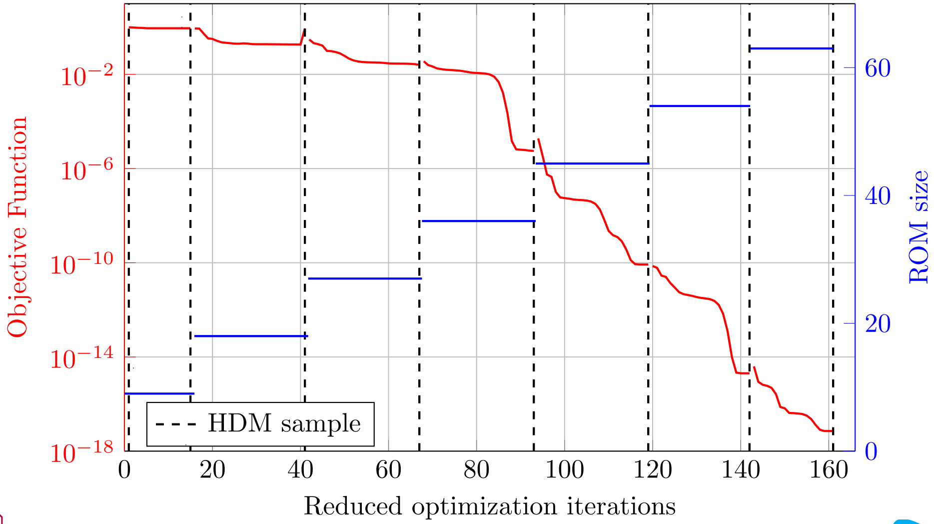
ROM-Constrained Optimization Solver Recovers Target



ROM Solver Requires 4× Fewer HDM Queries

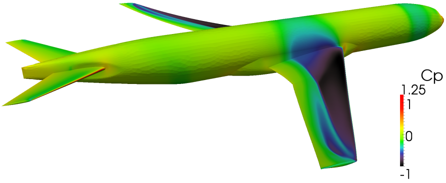


At the Cost of ROM Queries

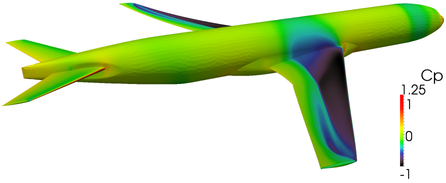


Next: Shape Optimization of Full Aircraft (CRM)

ROMs are fast, accurate, and require limited resources



HDM solution (Drag $= 142.336kN$)



ROM solution (Drag $= 142.304kN$)

- HDM: 70×10^6 DOF, **2hr on 1024** Intel Xeon E5-2698 v3 cores (2.3GHz)
- ROM: **170s on 2** Intel i7 cores (1.8GHz)
- Relative error in drag 0.022%
- CPU-time speedup greater than 2.15×10^4
- Wall-time speedup greater than 42
- *Washabaugh, Zahr, Farhat (AIAA, 2016)*

PDE-Constrained Optimization II

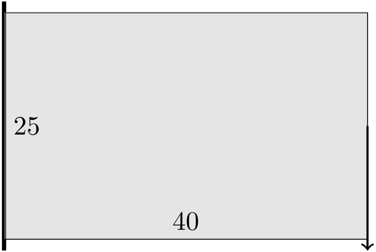
Goal: Rapidly solve PDE-constrained optimization problem of the form

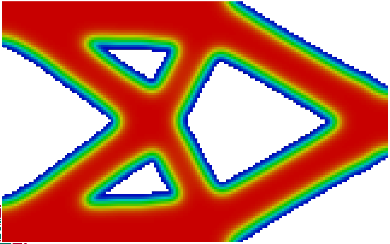
$$\begin{aligned} \underset{\boldsymbol{u}\in\mathbb{R}^{n_{\boldsymbol{u}}},\ \boldsymbol{\mu}\in\mathbb{R}^{n_{\boldsymbol{\mu}}}}{\text{minimize}} \quad & \mathcal{J}(\boldsymbol{u},\ \boldsymbol{\mu}) \\ \text{subject to} \quad & \boldsymbol{r}(\boldsymbol{u},\ \boldsymbol{\mu}) = 0 \\ & \boldsymbol{c}(\boldsymbol{u},\ \boldsymbol{\mu}) \geq 0 \end{aligned}$$

where

- $\boldsymbol{r} : \mathbb{R}^{n_{\boldsymbol{u}}} \times \mathbb{R}^{n_{\boldsymbol{\mu}}} \rightarrow \mathbb{R}^{n_{\boldsymbol{u}}}$ is the discretized partial differential equation
- $\mathcal{J} : \mathbb{R}^{n_{\boldsymbol{u}}} \times \mathbb{R}^{n_{\boldsymbol{\mu}}} \rightarrow \mathbb{R}$ is the objective function
- $\boldsymbol{c} : \mathbb{R}^{n_{\boldsymbol{u}}} \times \mathbb{R}^{n_{\boldsymbol{\mu}}} \rightarrow \mathbb{R}^{n_{\boldsymbol{c}}}$ are the side constraints
- $\boldsymbol{u} \in \mathbb{R}^{n_{\boldsymbol{u}}}$ is the PDE state vector
- $\boldsymbol{\mu} \in \mathbb{R}^{n_{\boldsymbol{\mu}}}$ is the vector of parameters

Problem Setup





- 16000 8-node brick elements, 77760 dofs
- Total Lagrangian form, finite strain, StVK[6]
- St. Venant-Kirchhoff material
- Sparse Cholesky linear solver (CHOLMOD[7])
- Newton-Raphson nonlinear solver
- Minimum compliance optimization problem

$$\begin{aligned} \underset{\boldsymbol{u}\in\mathbb{R}^{n_{\boldsymbol{u}}},\ \boldsymbol{\mu}\in\mathbb{R}^{n_{\boldsymbol{\mu}}}}{\text{minimize}} \quad & \boldsymbol{f}_{\text{ext}}{}^T\boldsymbol{u} \\ \text{subject to} \quad & V(\boldsymbol{\mu}) \leq \frac{1}{2}V_0 \\ & \boldsymbol{r}(\boldsymbol{u},\ \boldsymbol{\mu}) = 0 \end{aligned}$$

- Gradient computations: Adjoint method
- Optimizer: SNOPT [Gill et al., 2002]

[6] [Bonet and Wood, 1997, Belytschko et al., 2000]
[7] [Chen et al., 2008]

Restrict Parameter Space to Low-Dimensional Subspace

- *Restrict parameter to a low-dimensional subspace*

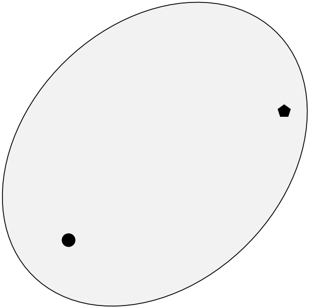
$$\boldsymbol{\mu} \approx \boldsymbol{\Phi}_{\boldsymbol{\mu}} \boldsymbol{\mu}_r$$

  - $\boldsymbol{\Phi}_{\boldsymbol{\mu}} = \begin{bmatrix} \boldsymbol{\phi}_{\boldsymbol{\mu}}^1 & \cdots & \boldsymbol{\phi}_{\boldsymbol{\mu}}^{k_{\boldsymbol{\mu}}} \end{bmatrix} \in \mathbb{R}^{n_{\boldsymbol{\mu}} \times k_{\boldsymbol{\mu}}}$ is the reduced basis
  - $\boldsymbol{\mu}_r \in \mathbb{R}^{k_{\boldsymbol{\mu}}}$ are the reduced coordinates of $\boldsymbol{\mu}$
  - $n_{\boldsymbol{\mu}} \gg k_{\boldsymbol{\mu}}$
- Substitute restriction into reduced-order model to obtain

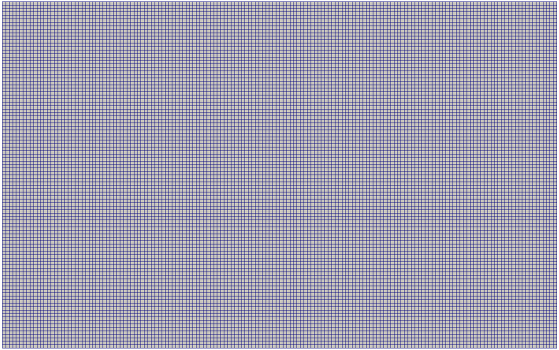
$$\boldsymbol{\Phi}_{\boldsymbol{u}}{}^T \boldsymbol{r}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r, \; \boldsymbol{\Phi}_{\boldsymbol{\mu}} \boldsymbol{\mu}_r) = 0$$

- Related work:
  [Maute and Ramm, 1995, Lieberman et al., 2010, Constantine et al., 2014]

Restrict Parameter Space to Low-Dimensional Subspace

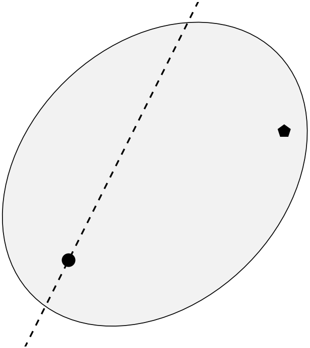


$\boldsymbol{\mu}$-space

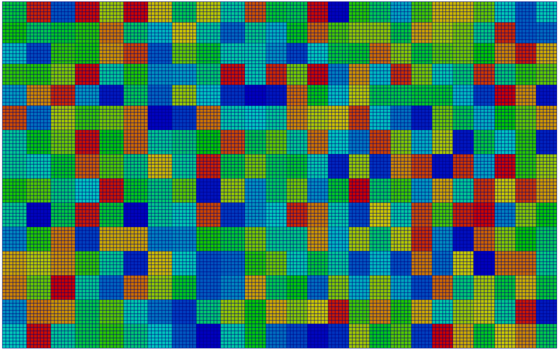


Background mesh

Restrict Parameter Space to Low-Dimensional Subspace



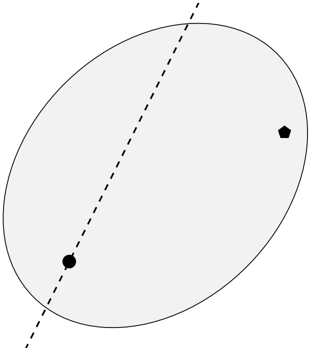
$\boldsymbol{\mu}$-space



Macroelements

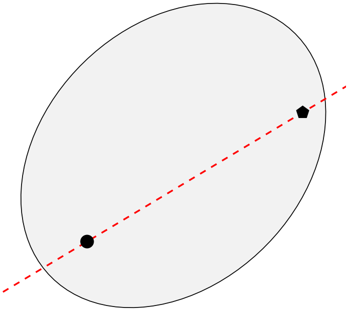
Optimality Conditions to Adapt Reduced-Order Basis, Φ_μ

- Selection of Φ_μ amounts to a *restriction* of the parameter space



Optimality Conditions to Adapt Reduced-Order Basis, Φ_μ

- Selection of Φ_μ amounts to a *restriction* of the parameter space
- Adaptation of Φ_μ should attempt to include the optimal solution in the restricted parameter space, i.e. $\mu^* \in \text{col}(\Phi_\mu)$
- Adaptation based on **first-order optimality conditions** of HDM optimization problem



Optimality Conditions to Adapt Reduced-Order Basis, $\boldsymbol{\Phi}_{\boldsymbol{\mu}}$

Lagrangian

$$\mathcal{L}(\boldsymbol{\mu},\ \boldsymbol{\lambda}) = \mathcal{J}(\boldsymbol{u}(\boldsymbol{\mu}),\ \boldsymbol{\mu}) - \boldsymbol{\lambda}^T \boldsymbol{c}(\boldsymbol{u}(\boldsymbol{\mu}),\ \boldsymbol{\mu})$$

Karush-Kuhn Tucker (KKT) Conditions[8]

$$\begin{aligned} \nabla_{\boldsymbol{\mu}} \mathcal{L}(\boldsymbol{\mu},\ \boldsymbol{\lambda}) &= 0 \\ \boldsymbol{\lambda} &\geq 0 \\ \boldsymbol{\lambda}_i \boldsymbol{c}_i(\boldsymbol{u}(\boldsymbol{\mu}),\ \boldsymbol{\mu}) &= 0 \\ \boldsymbol{c}(\boldsymbol{u}(\boldsymbol{\mu}), \boldsymbol{\mu}) &\geq 0 \end{aligned}$$

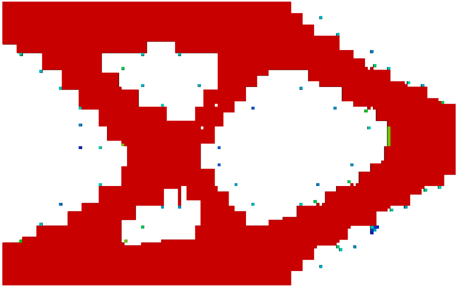
[8] [Nocedal and Wright, 2006]

Lagrangian Gradient Refinement Indicator

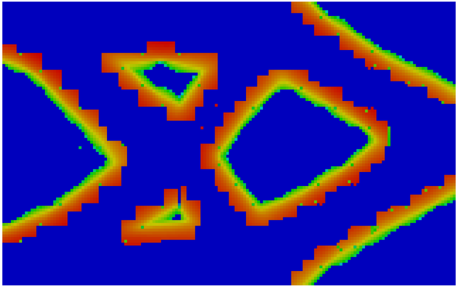
- From Lagrange multiplier estimates, only KKT condition not satisfied automatically:

$$\nabla_{\boldsymbol{\mu}}\mathcal{L}(\boldsymbol{\mu},\ \boldsymbol{\lambda}) = 0$$

- Use $|\nabla_{\boldsymbol{\mu}}\mathcal{L}(\boldsymbol{\mu},\ \boldsymbol{\lambda})|$ as indicator for **refinement** of discretization of $\boldsymbol{\mu}$-space



$\boldsymbol{\mu}$

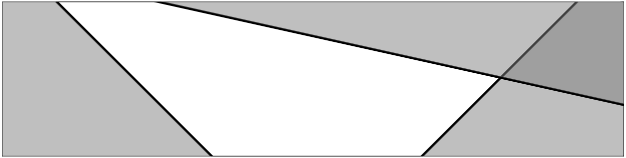


$|\nabla_{\boldsymbol{\mu}}\mathcal{L}(\boldsymbol{\mu},\ \boldsymbol{\lambda})|$

Constraints may lead to infeasible sub-problems

Non-Quadratic Trust-Region MOR [Zahr and Farhat, 2014]

$$\begin{aligned}
\underset{\boldsymbol{u}_r \in \mathbb{R}^{k_{\boldsymbol{u}}},\ \boldsymbol{\mu}_r \in \mathbb{R}^{k_{\boldsymbol{\mu}}}}{\text{minimize}} \quad & \mathcal{J}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}} \boldsymbol{\mu}_r) \\
\text{subject to} \quad & \boldsymbol{c}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}} \boldsymbol{\mu}_r) \geq 0 \\
& \boldsymbol{\Psi}_{\boldsymbol{u}}{}^T \boldsymbol{r}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}} \boldsymbol{\mu}_r) = 0 \\
& ||\boldsymbol{r}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}} \boldsymbol{\mu}_r)|| \leq \Delta
\end{aligned}$$



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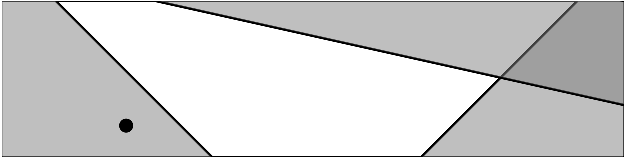
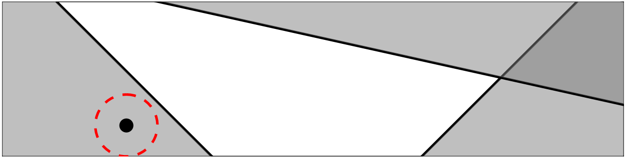


Constraints may lead to infeasible sub-problems

Non-Quadratic Trust-Region MOR [Zahr and Farhat, 2014]

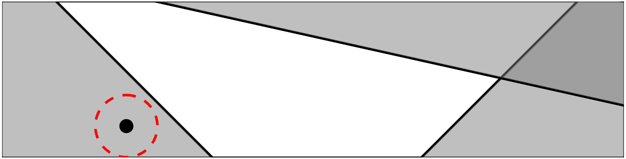
$$
\begin{aligned}
\underset{\boldsymbol{u}_r \in \mathbb{R}^{k_{\boldsymbol{u}}},\ \boldsymbol{\mu}_r \in \mathbb{R}^{k_{\boldsymbol{\mu}}}}{\text{minimize}} \quad & \mathcal{J}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}} \boldsymbol{\mu}_r) \\
\text{subject to} \quad & \boldsymbol{c}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}} \boldsymbol{\mu}_r) \geq 0 \\
& \boldsymbol{\Psi}_{\boldsymbol{u}}{}^T \boldsymbol{r}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}} \boldsymbol{\mu}_r) = 0 \\
& ||\boldsymbol{r}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}} \boldsymbol{\mu}_r)|| \leq \Delta
\end{aligned}
$$



Elastic constraints to circumvent infeasible subproblems

Constrained Non-Quadratic Trust-Region MOR (CNQTR-MOR)

$$\begin{aligned} \underset{\boldsymbol{u}_r\in\mathbb{R}^{k_{\boldsymbol{u}}},\ \boldsymbol{\mu}_r\in\mathbb{R}^{k_{\boldsymbol{\mu}}},\ \mathbf{t}\in\mathbb{R}^{n_{\boldsymbol{c}}}}{\text{minimize}} \quad & \mathcal{J}(\boldsymbol{\Phi}_{\boldsymbol{u}}\boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}}\boldsymbol{\mu}_r) - \gamma\mathbf{t}^T\mathbf{1} \\ \text{subject to} \quad & \boldsymbol{c}(\boldsymbol{\Phi}_{\boldsymbol{u}}\boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}}\boldsymbol{\mu}_r) \geq \mathbf{t} \\ & {\boldsymbol{\Psi}_{\boldsymbol{u}}}^T\boldsymbol{r}(\boldsymbol{\Phi}_{\boldsymbol{u}}\boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}}\boldsymbol{\mu}_r) = 0 \\ & ||\boldsymbol{r}(\boldsymbol{\Phi}_{\boldsymbol{u}}\boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}}\boldsymbol{\mu}_r)|| \leq \Delta \\ & \mathbf{t} \leq 0 \end{aligned}$$



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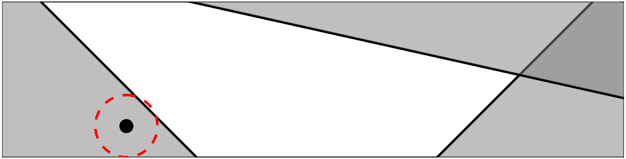
\text{subject to} \quad & \boldsymbol{c}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}} \boldsymbol{\mu}_r) \geq \mathbf{t} \\
& \boldsymbol{\Psi}_{\boldsymbol{u}}{}^T \boldsymbol{r}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}} \boldsymbol{\mu}_r) = 0 \\
& ||\boldsymbol{r}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}} \boldsymbol{\mu}_r)|| \leq \Delta \\
& \mathbf{t} \leq 0
\end{aligned}$$



Elastic constraints to circumvent infeasible subproblems

Constrained Non-Quadratic Trust-Region MOR (CNQTR-MOR)

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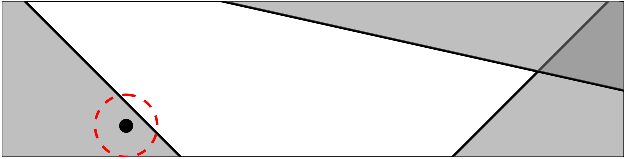
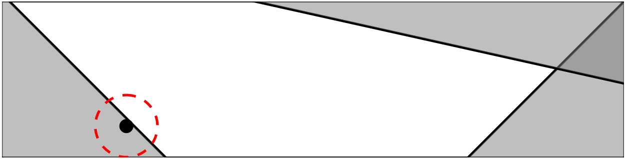
\underset{\boldsymbol{u}_r\in\mathbb{R}^{k_{\boldsymbol{u}}},\ \boldsymbol{\mu}_r\in\mathbb{R}^{k_{\boldsymbol{\mu}}},\ \mathbf{t}\in\mathbb{R}^{n_{\boldsymbol{c}}}}{\text{minimize}} \quad & \mathcal{J}(\boldsymbol{\Phi}_{\boldsymbol{u}}\boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}}\boldsymbol{\mu}_r) - \gamma\mathbf{t}^T\mathbf{1} \\
\text{subject to} \quad & \boldsymbol{c}(\boldsymbol{\Phi}_{\boldsymbol{u}}\boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}}\boldsymbol{\mu}_r) \geq \mathbf{t} \\
& \boldsymbol{\Psi}_{\boldsymbol{u}}{}^T\boldsymbol{r}(\boldsymbol{\Phi}_{\boldsymbol{u}}\boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}}\boldsymbol{\mu}_r) = 0 \\
& ||\boldsymbol{r}(\boldsymbol{\Phi}_{\boldsymbol{u}}\boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}}\boldsymbol{\mu}_r)|| \leq \Delta \\
& \mathbf{t} \leq 0
\end{aligned}$$



Elastic constraints to circumvent infeasible subproblems

Constrained Non-Quadratic Trust-Region MOR (CNQTR-MOR)

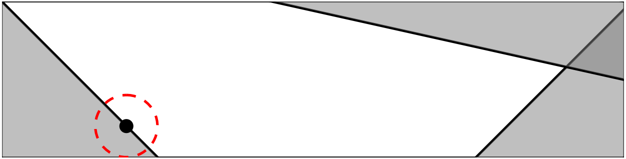
$$\begin{aligned}
\underset{\boldsymbol{u}_r\in\mathbb{R}^{k_{\boldsymbol{u}}},\ \boldsymbol{\mu}_r\in\mathbb{R}^{k_{\boldsymbol{\mu}}},\ \mathbf{t}\in\mathbb{R}^{n_{\boldsymbol{c}}}}{\text{minimize}} \quad & \mathcal{J}(\boldsymbol{\Phi}_{\boldsymbol{u}}\boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}}\boldsymbol{\mu}_r) - \gamma\mathbf{t}^T\mathbf{1} \\
\text{subject to} \quad & \boldsymbol{c}(\boldsymbol{\Phi}_{\boldsymbol{u}}\boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}}\boldsymbol{\mu}_r) \geq \mathbf{t} \\
& \boldsymbol{\Psi}_{\boldsymbol{u}}{}^T\boldsymbol{r}(\boldsymbol{\Phi}_{\boldsymbol{u}}\boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}}\boldsymbol{\mu}_r) = 0 \\
& ||\boldsymbol{r}(\boldsymbol{\Phi}_{\boldsymbol{u}}\boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}}\boldsymbol{\mu}_r)|| \leq \Delta \\
& \mathbf{t} \leq 0
\end{aligned}$$



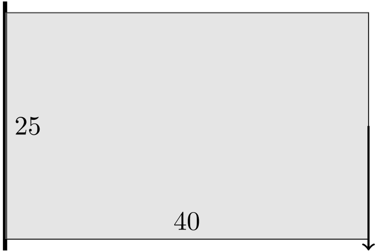
Elastic constraints to circumvent infeasible subproblems

Constrained Non-Quadratic Trust-Region MOR (CNQTR-MOR)

$$\begin{aligned}
\underset{\boldsymbol{u}_r \in \mathbb{R}^{k_{\boldsymbol{u}}},\ \boldsymbol{\mu}_r \in \mathbb{R}^{k_{\boldsymbol{\mu}}},\ \mathbf{t} \in \mathbb{R}^{n_{\boldsymbol{c}}}}{\text{minimize}} \quad & \mathcal{J}(\boldsymbol{\Phi}_{\boldsymbol{u}}\boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}}\boldsymbol{\mu}_r) - \gamma \mathbf{t}^T \mathbf{1} \\
\text{subject to} \quad & \boldsymbol{c}(\boldsymbol{\Phi}_{\boldsymbol{u}}\boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}}\boldsymbol{\mu}_r) \geq \mathbf{t} \\
& \boldsymbol{\Psi}_{\boldsymbol{u}}{}^T \boldsymbol{r}(\boldsymbol{\Phi}_{\boldsymbol{u}}\boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}}\boldsymbol{\mu}_r) = 0 \\
& ||\boldsymbol{r}(\boldsymbol{\Phi}_{\boldsymbol{u}}\boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}}\boldsymbol{\mu}_r)|| \leq \Delta \\
& \mathbf{t} \leq 0
\end{aligned}$$



Compliance Minimization: 2D Cantilever



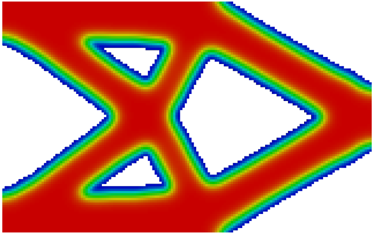
- 16000 8-node brick elements, 77760 dofs
- Total Lagrangian form, finite strain, StVK[9]
- St. Venant-Kirchhoff material
- Sparse Cholesky linear solver (CHOLMOD[10])
- Newton-Raphson nonlinear solver
- Minimum compliance optimization problem

$$\begin{aligned} \underset{\boldsymbol{u}\in\mathbb{R}^{n_{\boldsymbol{u}}},\ \boldsymbol{\mu}\in\mathbb{R}^{n_{\boldsymbol{\mu}}}}{\text{minimize}} \quad & \boldsymbol{f}_{\text{ext}}{}^T\boldsymbol{u} \\ \text{subject to} \quad & V(\boldsymbol{\mu}) \leq \frac{1}{2}V_0 \\ & \boldsymbol{r}(\boldsymbol{u},\ \boldsymbol{\mu}) = 0 \end{aligned}$$

- Gradient computations: Adjoint method
- Optimizer: SNOPT [Gill et al., 2002]
- Maximum ROM size: $k_{\boldsymbol{u}} \leq 5$

[9] [Bonet and Wood, 1997, Belytschko et al., 2000]
[10] [Chen et al., 2008]

Order of Magnitude Speedup to Suboptimal Solution



HDM

CNQTR-MOR + Φ_μ adaptivity

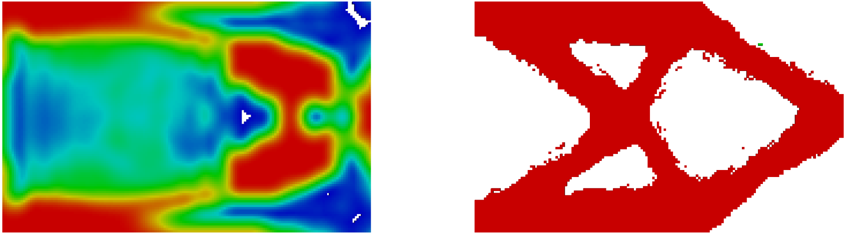
HDM Solution	**HDM** Gradient	**HDM** Optimization
7458s (450)	4018s (411)	8284s

HDM
Elapsed time = 19761s

HDM Solution	**HDM** Gradient	**ROB** Construction	**ROM** Optimization
1049s (64)	88s (9)	727s (56)	39s (3676)

CNQTR-MOR + Φ_μ adaptivity
Elapsed time = 2197s, Speedup ≈ 9x

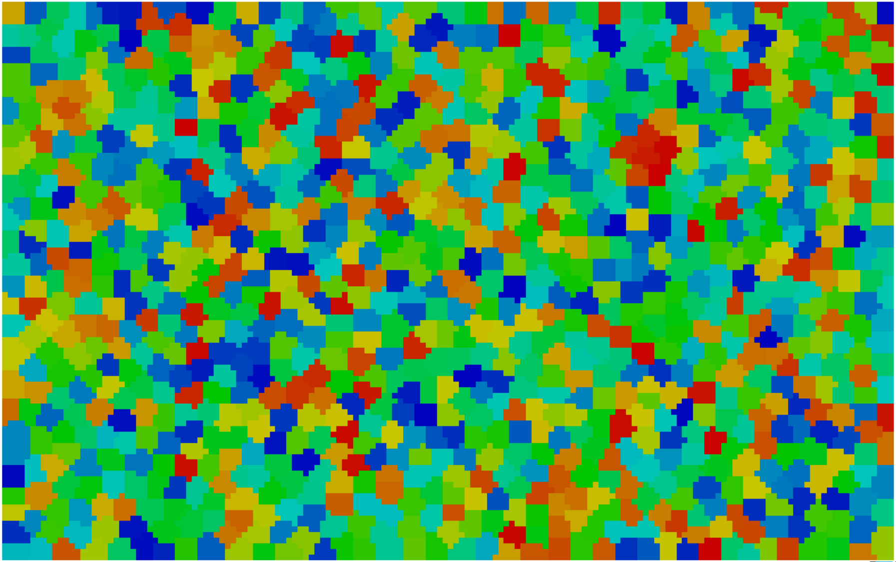
Better Solution after 64 HDM Evaluations



HDM CNQTR-MOR + $\boldsymbol{\Phi}_\mu$ adaptivity

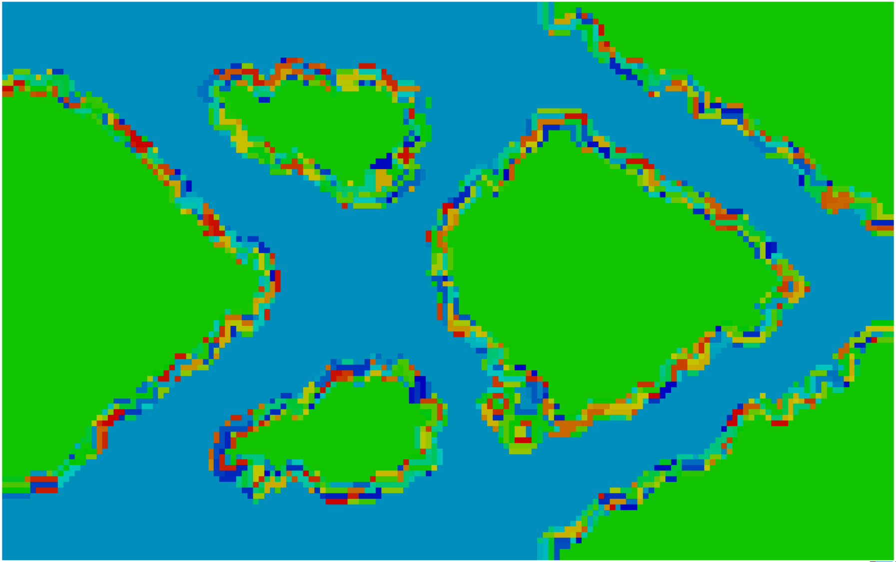
- CNQTR-MOR + $\boldsymbol{\Phi}_\mu$ adaptivity: superior approximation to optimal solution than HDM approach after fixed number of HDM solves (64)
- Reasonable option to *warm-start* HDM topology optimization

Macro-element Evolution



Iteration 0 (1000)

Macro-element Evolution



Iteration 1 (977)

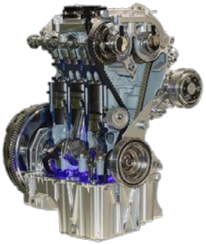
CNQTR-MOR + Φ_μ adaptivity

An Adaptive Reduction Framework for Optimization under Uncertainty

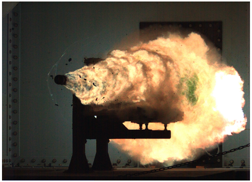
- Highly volatile systems tend to be plagued by uncertainties, which must be quantified for meaningful problem formulation
- Optimize *moments* of quantities of interest of stochastic partial differential equation

$$\begin{aligned} \underset{\boldsymbol{u}\in\mathbb{R}^{n_{\boldsymbol{u}}},\ \boldsymbol{\mu}\in\mathbb{R}^{n_{\boldsymbol{\mu}}}}{\text{minimize}} \quad & \int_{\boldsymbol{\Xi}} \mathcal{J}(\boldsymbol{u},\ \boldsymbol{\mu};\ \boldsymbol{\xi})\, d\boldsymbol{\xi} \\ \text{subject to} \quad & \boldsymbol{r}(\boldsymbol{u},\ \boldsymbol{\mu};\ \boldsymbol{\xi}) = 0 \qquad \boldsymbol{\xi} \in \boldsymbol{\Xi} \end{aligned}$$

- Combine adaptive model reduction framework with dimension-adaptive sparse grids to **enable** stochastic optimization



Engine System



EM Launcher

Collaborators: Drew Kouri (Sandia NM), Kevin Carlberg (Sandia CA)

High-Order Methods for Optimization of Conservation Laws

- Derived, implemented fully discrete adjoint method for globally high-order discretization of conservation laws on deforming domains
- *Incorporation of time-periodicity constraints*

Initial	Optimal Control	Optimal Shape/Control
Energy = 9.4096e+00 Thrust = 1.7660e-01	Energy = 4.9476e+00 Thrust = 2.5000e+00	Energy = 4.6110e+00 Thrust = 2.5000e+00

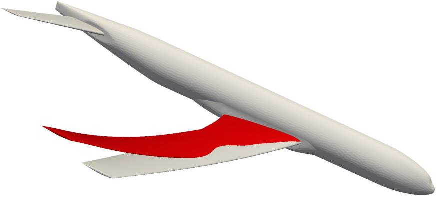
Initial | Optimal Control | Optimal Shape/Control

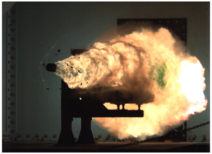
Collaborators: Per-Olof Persson (UCB, LBNL), Jon Wilkening (UCB, LBNL)

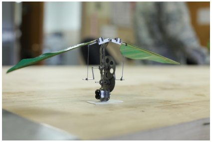
Approaching Many-Query, Extreme-Scale Computational Physics

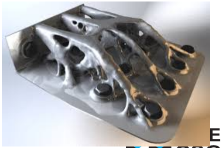
Leveraging Inexactness For Acceleration of Many-Query Multiphysics Problems

- Framework introduced for accelerating PDE-constrained optimization problems with **side constraints** and **large-dimensional parameter space**
  - Adaptive reduction of state and parameter spaces
- Applied to aerodynamic design and topology optimization
  - Order of magnitude speedup speedup observed
  - Competitive warm-start method

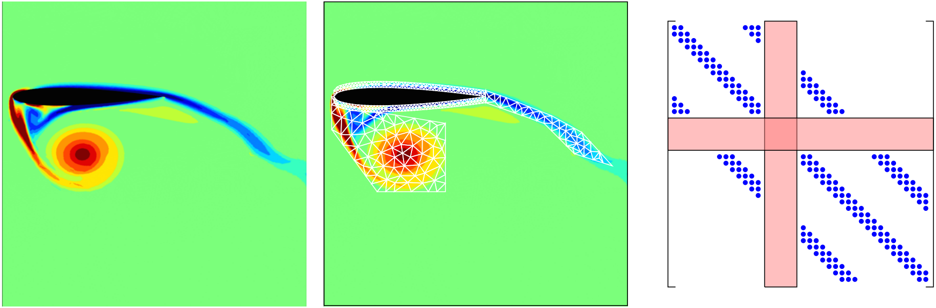








Faster Computational Physics: Adaptive Data-Driven Discretization



(a) Vorticity around heaving airfoil (b) Potential Ω^l, Ω^g decomposition (c) Idealized sparsity structure

- Methods to *transform* features in global basis functions - minimize reliance on local shape functions
- Linear algebra for sparse operators with a few dense rows and columns
- Elements of: **high-order methods**, **adaptive mesh refinement**, **numerical linear algebra**

Fewer Queries: Second-Order Methods for Accelerated Convergence

Hessian information highly desired in optimization and UQ, but expensive due to $\mathcal{O}(N_{\boldsymbol{\mu}})$ required linear system solves

Sensitivity/Adjoint Method for Computing Hessian

$$\frac{\mathrm{d}^2\mathcal{J}}{\mathrm{d}\boldsymbol{\mu}_j \mathrm{d}\boldsymbol{\mu}_k} = \frac{\partial^2\mathcal{J}}{\partial\boldsymbol{\mu}_j\partial\boldsymbol{\mu}_k} + \frac{\partial^2\mathcal{J}}{\partial\boldsymbol{\mu}_j\partial\boldsymbol{u}}\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{\mu}_k} + \frac{\partial\boldsymbol{u}}{\partial\boldsymbol{\mu}_j}^T\frac{\partial^2\mathcal{J}}{\partial\boldsymbol{u}\partial\boldsymbol{\mu}_k} + \frac{\partial\boldsymbol{u}}{\partial\boldsymbol{\mu}_j}^T\frac{\partial^2\mathcal{J}}{\partial\boldsymbol{u}\partial\boldsymbol{u}}\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{\mu}_k}$$
$$- \frac{\partial\mathcal{J}}{\partial\boldsymbol{u}}\frac{\partial\boldsymbol{r}}{\partial\boldsymbol{u}}^{-1}\left[\frac{\partial^2\boldsymbol{r}}{\partial\boldsymbol{\mu}_j\partial\boldsymbol{\mu}_k} + \frac{\partial^2\boldsymbol{r}}{\partial\boldsymbol{\mu}_j\partial\boldsymbol{u}}\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{\mu}_k} + \frac{\partial^2\boldsymbol{r}}{\partial\boldsymbol{\mu}_k\partial\boldsymbol{u}}\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{\mu}_j} + \frac{\partial^2\boldsymbol{r}}{\partial\boldsymbol{u}\partial\boldsymbol{u}} : \frac{\partial\boldsymbol{u}}{\partial\boldsymbol{\mu}_j}\otimes\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{\mu}_k}\right]$$

where

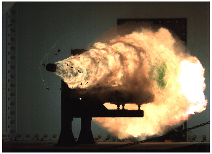
$$\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{\mu}_j} = \frac{\partial\boldsymbol{r}}{\partial\boldsymbol{u}}^{-1}\frac{\partial\boldsymbol{r}}{\partial\boldsymbol{\mu}_j}$$

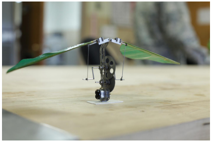
- Fast, *multiple right-hand side* linear solver by building data-driven subspace for image of $\frac{\partial\boldsymbol{r}}{\partial\boldsymbol{u}}^{-1}, \frac{\partial\boldsymbol{r}}{\partial\boldsymbol{u}}^{-T}$
- MOR concepts in context of **numerical linear algebra**

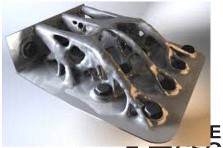
Approaching Many-Query, Extreme-Scale Computational Physics

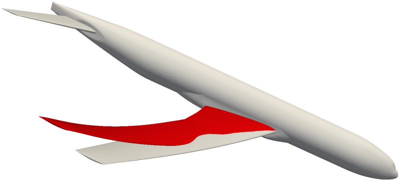
Leveraging Inexactness For Acceleration of Many-Query Multiphysics Problems

- Framework introduced for accelerating PDE-constrained optimization problems with **side constraints** and **large-dimensional parameter space**
  - Adaptive reduction of state and parameter spaces
- Applied to aerodynamic design and topology optimization
  - Order of magnitude speedup speedup observed
  - Competitive warm-start method
- **Future work:** combine advantages of MOR/AMR for drastic computational savings with *in-situ* training; second-order methods for rapidly converging many-query algorithms; new (multiphysics) applications









Acknowledgement

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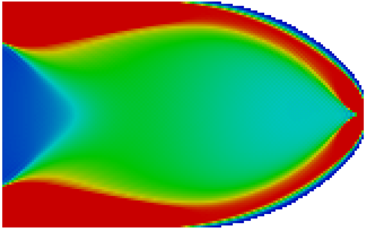
Standard Difficulty: Binary Solutions



(a) Without penalization

Standard Difficulty: Binary Solutions

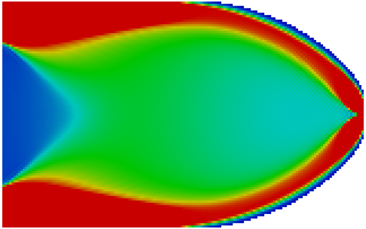
Relaxed, Penalized Problem Setup

$$\begin{aligned} \underset{\boldsymbol{u}\in\mathbb{R}^{n_{\boldsymbol{u}}},\ \boldsymbol{\mu}\in\mathbb{R}^{n_{\boldsymbol{\mu}}}}{\text{minimize}} \quad & \boldsymbol{f}_{\text{ext}}{}^T\boldsymbol{u} \\ \text{subject to} \quad & V(\boldsymbol{\mu}) \leq \frac{1}{2}V_0 \\ & \mathbf{r}(\boldsymbol{u},\ \boldsymbol{\mu}^p) = 0 \\ & \boldsymbol{\mu} \in [0,1]^{k_{\boldsymbol{\mu}}} \end{aligned}$$

Effect of Penalization

$$\mathbf{K}^e \leftarrow (\boldsymbol{\mu}^e)^p\mathbf{K}^e$$

- $\mathbf{K}^e$: eth element stiffness matrix



(a) Without penalization

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Effect of Penalization

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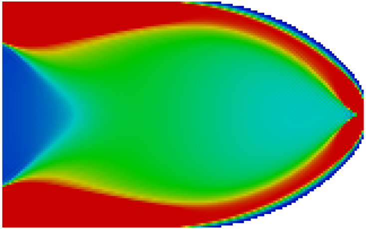
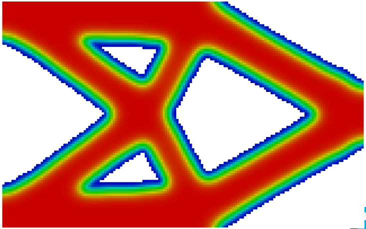


(a) Without penalization

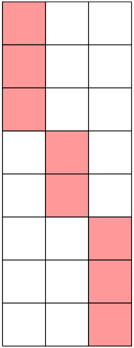


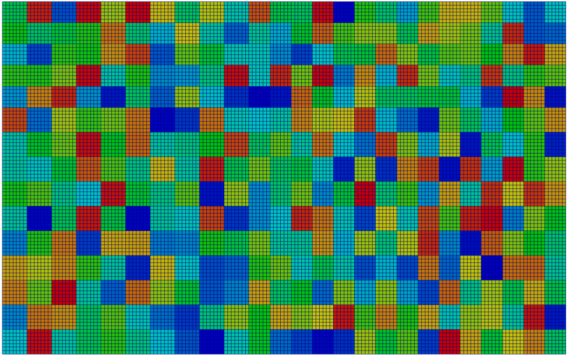
(b) With penalization

Standard Difficulty: Binary Solutions

Implication for ROM

- From parameter restriction, $\boldsymbol{\mu}^p = (\boldsymbol{\Phi}_{\boldsymbol{\mu}} \boldsymbol{\mu}_r)^p$
- Precomputation relies on separability of $\boldsymbol{\Phi}_{\boldsymbol{\mu}}$ and $\boldsymbol{\mu}_r$
- Separability maintained if $(\boldsymbol{\Phi}_{\boldsymbol{\mu}} \boldsymbol{\mu}_r)^p = \boldsymbol{\Phi}_{\boldsymbol{\mu}} \boldsymbol{\mu}_r^p$
- Sufficient condition: *columns of* $\boldsymbol{\Phi}_{\boldsymbol{\mu}}$ *have non-overlapping non-zeros*





Efficient Evaluation of Nonlinear Terms

- Due to the mixing of high-dimensional and low-dimensional terms in the ROM expression, only limited speedups available

$$\mathbf{r}_r(\boldsymbol{u}_r,\ \boldsymbol{\mu}_r) = \boldsymbol{\Phi}_{\boldsymbol{u}}{}^T \mathbf{r}(\boldsymbol{\Phi}_{\boldsymbol{u}} \boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}} \boldsymbol{\mu}_r) = 0$$

- To enable *pre-computation* of all large-dimensional quantities into low-dimensional ones, leverage *Taylor series expansion*

$$\begin{aligned}[\mathbf{r}_r(\boldsymbol{u}_r,\ \boldsymbol{\mu}_r)]_i = \mathbf{D}^0_{im}(\boldsymbol{\mu}_r)_m + \mathbf{D}^1_{ijm}(\boldsymbol{u}_r \times \boldsymbol{\mu}_r)_{jm} + \mathbf{D}^2_{ijkm}(\boldsymbol{u}_r \times \boldsymbol{u}_r \times \boldsymbol{\mu}_r)_{jkm} \\ + \mathbf{D}^3_{ijklm}(\boldsymbol{u}_r \times \boldsymbol{u}_r \times \boldsymbol{u}_r \times \boldsymbol{\mu}_r)_{jklm} = 0\end{aligned}$$

where

$$\mathbf{D}^3_{ijklm} = \frac{\partial^3 \mathbf{r}_t}{\partial \boldsymbol{u}_p \partial \boldsymbol{u}_q \partial \boldsymbol{u}_s}(\hat{\boldsymbol{u}},\ \boldsymbol{\phi}^m_{\boldsymbol{\mu}})(\boldsymbol{\phi}^i_{\boldsymbol{u}} \times \boldsymbol{\phi}^j_{\boldsymbol{u}} \times \boldsymbol{\phi}^k_{\boldsymbol{u}} \times \boldsymbol{\phi}^l_{\boldsymbol{u}})_{tpqs}$$

- Related work: [Rewienski, 2003, Barrault et al., 2004, Barbič and James, 2007, Nguyen and Peraire, 2008, Chaturantabut and Sorensen, 2010, Carlberg et al., 2011]

Lagrange Multiplier Estimate

Lagrange Multiplier, Constraint Pairs

$\boldsymbol{\lambda}$	$\boldsymbol{\lambda}_r$	$\boldsymbol{\tau}$	$\boldsymbol{\tau}_r$
$\mathbf{c}(\boldsymbol{u},\ \boldsymbol{\mu}) \geq 0$	$\mathbf{c}(\boldsymbol{\Phi}_{\boldsymbol{u}}\boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}}\boldsymbol{\mu}) \geq 0$	$\mathbf{A}\boldsymbol{\mu} \geq \mathbf{b}$	$\mathbf{A}_r\boldsymbol{\mu}_r \geq \mathbf{b}_r$

Goal: Given $\boldsymbol{u}_r,\ \boldsymbol{\mu}_r,\ \boldsymbol{\tau}_r \geq 0,\ \boldsymbol{\lambda}_r \geq 0$, estimate $\tilde{\boldsymbol{\tau}} \geq 0,\ \tilde{\boldsymbol{\lambda}} \geq 0$ to compute

$$\nabla_{\boldsymbol{\mu}}\mathcal{L}(\boldsymbol{\Phi}_{\boldsymbol{\mu}}\boldsymbol{\mu}_r,\ \tilde{\boldsymbol{\lambda}},\ \tilde{\boldsymbol{\tau}}) = \frac{\partial \mathcal{J}}{\partial \boldsymbol{\mu}}(\boldsymbol{\Phi}_{\boldsymbol{u}}\boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}}\boldsymbol{\mu}_r) - \frac{\partial \mathbf{c}}{\partial \boldsymbol{\mu}}(\boldsymbol{\Phi}_{\boldsymbol{u}}\boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}}\boldsymbol{\mu}_r)^T\tilde{\boldsymbol{\lambda}} - \mathbf{A}^T\tilde{\boldsymbol{\tau}}$$

Lagrange Multiplier Estimates

$$\tilde{\boldsymbol{\lambda}} = \boldsymbol{\lambda}_r$$

$$\tilde{\boldsymbol{\tau}} = \underset{\boldsymbol{\tau}\geq 0}{\arg\min} \left|\left|\mathbf{A}^T\boldsymbol{\tau} - \left(\frac{\partial \mathcal{J}}{\partial \boldsymbol{\mu}}(\boldsymbol{\Phi}_{\boldsymbol{u}}\boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}}\boldsymbol{\mu}_r) - \frac{\partial \mathbf{c}}{\partial \boldsymbol{\mu}}(\boldsymbol{\Phi}_{\boldsymbol{u}}\boldsymbol{u}_r,\ \boldsymbol{\Phi}_{\boldsymbol{\mu}}\boldsymbol{\mu}_r)^T\tilde{\boldsymbol{\lambda}}\right)\right|\right|$$

Non-negative least squares: [Lawson and Hanson, 1974, Chapman et al., 2015]

Standard Difficulty: Checkerboarding

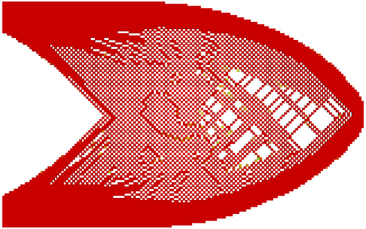
Gradient Filtering, Nodal Projection

- Minimum length scale, $r_{\min}$
- Gradient Filtering[11]

$$\widehat{\frac{\partial \mathcal{J}}{\partial \boldsymbol{\mu}_k}} = \frac{\sum_{j \in S_k} H_{kj} \boldsymbol{\mu}_i \frac{\partial \mathcal{J}}{\partial \boldsymbol{\mu}_i}}{\boldsymbol{\mu}_k \sum_{j \in S_k} H_{kj}}$$

- Nodal Projection

$$\boldsymbol{\mu}_k = \frac{\sum_{j \in \mathcal{S}_k} \boldsymbol{\tau}_j H_{jk}}{\sum_{j \in \mathcal{S}_k} H_{jk}}$$



(a) Without projection/filtering

[11] $H_{ki} = r_{\min} - \mathrm{dist}(k,\ i)$

Standard Difficulty: Checkerboarding

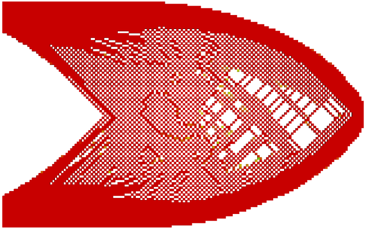
Gradient Filtering, Nodal Projection

- Minimum length scale, $r_{\min}$
- Gradient Filtering[11]

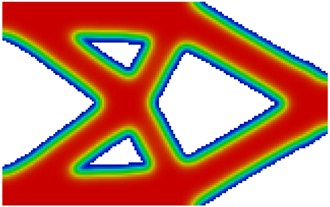
$$\widehat{\frac{\partial \mathcal{J}}{\partial \boldsymbol{\mu}_k}} = \frac{\sum_{j \in S_k} H_{kj} \boldsymbol{\mu}_i \frac{\partial \mathcal{J}}{\partial \boldsymbol{\mu}_i}}{\boldsymbol{\mu}_k \sum_{j \in S_k} H_{kj}}$$

- Nodal Projection

$$\boldsymbol{\mu}_k = \frac{\sum_{j \in \mathcal{S}_k} \boldsymbol{\tau}_j H_{jk}}{\sum_{j \in \mathcal{S}_k} H_{jk}}$$



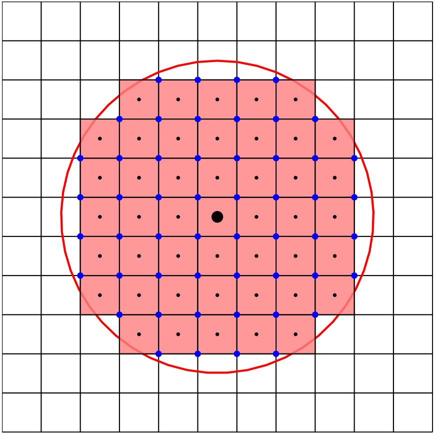
(a) Without projection/filtering



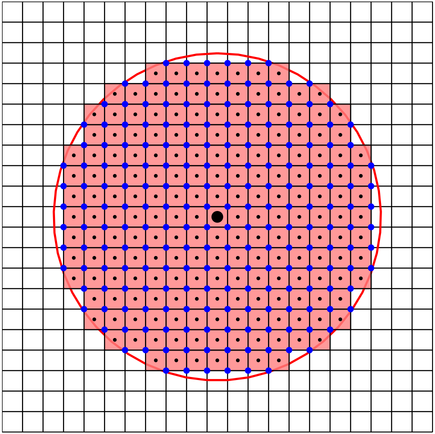
(b) With projection

[11] $H_{ki} = r_{\min} - \mathrm{dist}(k,\ i)$

Standard Difficulty: Checkerboarding



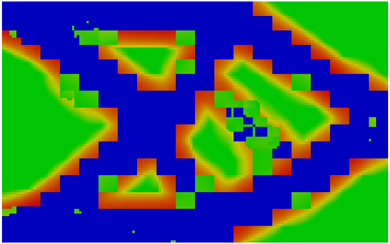
Standard Difficulty: Checkerboarding



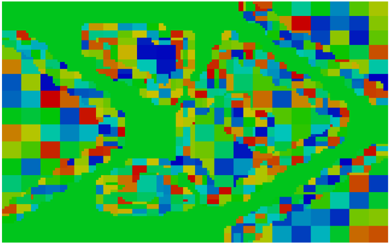
Standard Difficulty: Checkerboarding

Implication for ROM

- Nonlocality introduced through projection/filtering
- μ_e influences volume fraction of all elements within r_{min} of element/node e
- Clashes with requirement on Φ_μ of columns with non-overlapping non-zeros
- Handled heuristically by performing parameter basis adaptation to eliminate "checkerboard" regions of parameter space, uses concept of r_{min}
- *Next: Helmholtz filtering*



Gradient of Lagrangian



Updated Macroelements

Standard Difficulty: Checkerboarding

Implication for ROM

- Nonlocality introduced through projection/filtering
- $\boldsymbol{\mu}_e$ influences volume fraction of all elements within $r_{\min}$ of element/node e
- Clashes with requirement on $\boldsymbol{\Phi}_{\boldsymbol{\mu}}$ of columns with non-overlapping non-zeros
- Handled heuristically by performing parameter basis adaptation to eliminate "checkerboard" regions of parameter space, uses concept of $r_{\min}$
- *Next: Helmholtz filtering*