Energetically Optimal Flapping Flight via a Fully Discrete Adjoint Method with Explicit Treatment of Flapping Frequency

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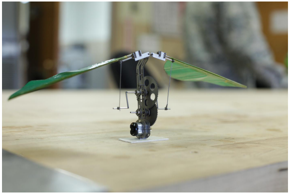
Understand and design energetically optimal flapping motions

Energetically optimal flapping flight critical to

- understand biological systems
- design Micro Aerial Vehicles (MAVs)



Optimal flapping motion of micro aerial vehicle

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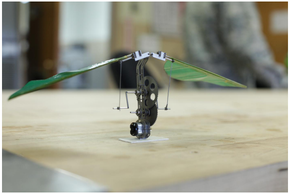


Optimal flapping motion of micro aerial vehicle

Flapping frequency critical consideration in energetically optimal flapping

Challenge: Parametrize frequency $\Longrightarrow$ parametrize time domain

- N_t uniform timesteps per period required for accuracy
- Flapping frequency (period) is parametrized $f = f(\boldsymbol{\mu})$ $(T = T(\boldsymbol{\mu}))$

$$T(\boldsymbol{\mu}) = N_t \Delta t$$

Challenge: Parametrize frequency $\implies$ parametrize time domain

- N_t uniform timesteps per period required for accuracy
- Flapping frequency (period) is parametrized $f = f(\boldsymbol{\mu})$ $(T = T(\boldsymbol{\mu}))$

$$T(\boldsymbol{\mu}) = N_t \Delta t$$

Fix N_t*, parametrize* $\Delta t = \Delta t(\boldsymbol{\mu})$

Generalization beyond flapping

Generalization: PDE-constrained optimization with parametrized time domain

- Optimal control
- Determination of fundamental frequency, e.g., von Karman vortex shedding
- **Path/trajectory optimization**: find motion that achieves desired final position in least amount of time

Unsteady PDE-constrained optimization formulation

Goal: Find the solution of the *unsteady PDE-constrained optimization* problem

$$\begin{aligned} \underset{\boldsymbol{U},\ \boldsymbol{\mu}}{\text{minimize}} \quad & \mathcal{J}(\boldsymbol{U}, \boldsymbol{\mu}) \\ \text{subject to} \quad & \boldsymbol{C}(\boldsymbol{U}, \boldsymbol{\mu}) \leq 0 \\ & \frac{\partial \boldsymbol{U}}{\partial t} + \nabla \cdot \boldsymbol{F}(\boldsymbol{U}, \nabla \boldsymbol{U}) = 0 \quad \text{in} \quad v(\boldsymbol{\mu}, t) \end{aligned}$$

where $t \in [0,\ T(\boldsymbol{\mu})]$ and

- $\boldsymbol{U}(\boldsymbol{x}, t)$ PDE solution
- $\boldsymbol{\mu}$ design/control parameters
- $\mathcal{J}(\boldsymbol{U}, \boldsymbol{\mu}) = \dfrac{1}{T(\boldsymbol{\mu})} \displaystyle\int_0^{T(\boldsymbol{\mu})} \int_{\boldsymbol{\Gamma}} j(\boldsymbol{U}, \boldsymbol{\mu}, t)\, dS\, dt$ objective function
- $\boldsymbol{C}(\boldsymbol{U}, \boldsymbol{\mu}) = \dfrac{1}{T(\boldsymbol{\mu})} \displaystyle\int_0^{T(\boldsymbol{\mu})} \int_{\boldsymbol{\Gamma}} \mathbf{c}(\boldsymbol{U}, \boldsymbol{\mu}, t)\, dS\, dt$ constraints

Nested approach to PDE-constrained optimization

PDE optimization requires repeated queries to primal and dual PDE

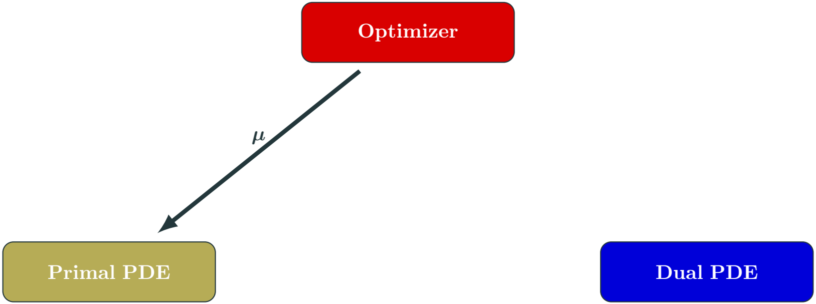
Optimizer

Primal PDE

Dual PDE

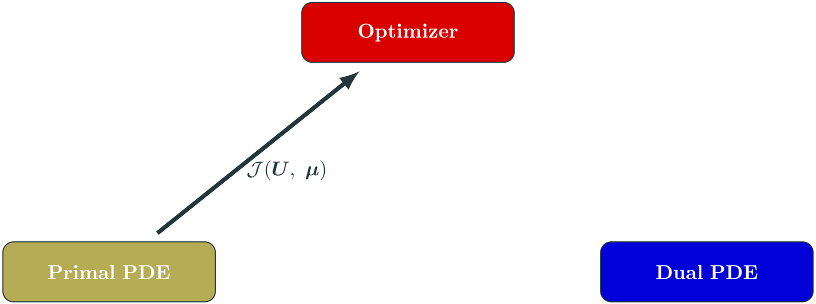
Nested approach to PDE-constrained optimization

PDE optimization requires repeated queries to primal and dual PDE



Nested approach to PDE-constrained optimization

PDE optimization requires repeated queries to primal and dual PDE



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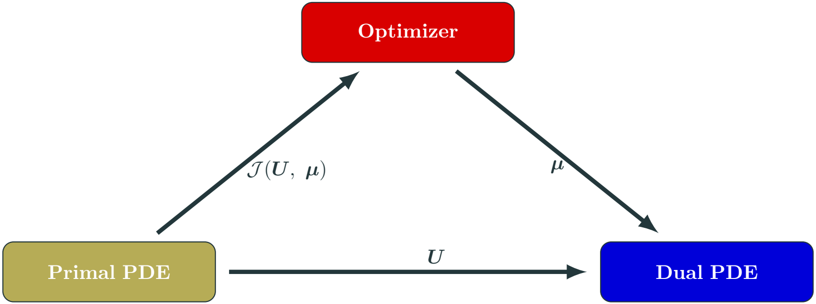
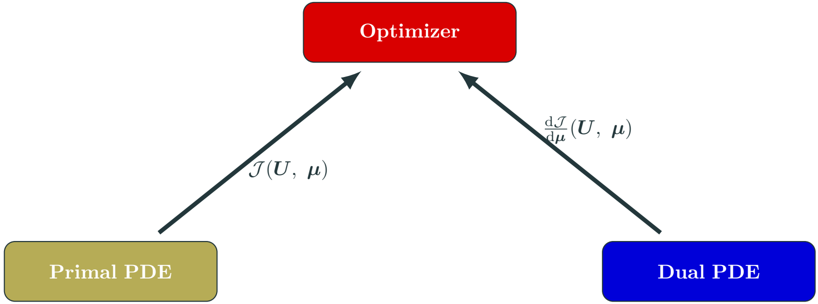
PDE optimization requires repeated queries to primal and dual PDE



Nested approach to PDE-constrained optimization

PDE optimization requires repeated queries to primal and dual PDE



High-order discretization of PDE-constrained optimization

- *Continuous* PDE-constrained optimization problem

$$\begin{aligned} \underset{\boldsymbol{U},\ \boldsymbol{\mu}}{\text{minimize}} \quad & \mathcal{J}(\boldsymbol{U}, \boldsymbol{\mu}) \\ \text{subject to} \quad & \boldsymbol{C}(\boldsymbol{U}, \boldsymbol{\mu}) \leq 0 \\ & \frac{\partial \boldsymbol{U}}{\partial t} + \nabla \cdot \boldsymbol{F}(\boldsymbol{U}, \nabla \boldsymbol{U}) = 0 \quad \text{in} \quad v(\boldsymbol{\mu}, t) \end{aligned}$$

- *Fully discrete* PDE-constrained optimization problem

$$\begin{aligned} \underset{\substack{\boldsymbol{u}_0,\ \ldots,\ \boldsymbol{u}_{N_t} \in \mathbb{R}^{N_{\boldsymbol{u}}}, \\ \boldsymbol{k}_{1,1},\ \ldots,\ \boldsymbol{k}_{N_t,s} \in \mathbb{R}^{N_{\boldsymbol{u}}}, \\ \boldsymbol{\mu} \in \mathbb{R}^{n_{\boldsymbol{\mu}}}}}{\text{minimize}} \quad & J(\boldsymbol{u}_0,\ \ldots,\ \boldsymbol{u}_{N_t},\ \boldsymbol{k}_{1,1},\ \ldots,\ \boldsymbol{k}_{N_t,s},\ \boldsymbol{\mu}) \\ \text{subject to} \quad & \mathbf{C}(\boldsymbol{u}_0,\ \ldots,\ \boldsymbol{u}_{N_t},\ \boldsymbol{k}_{1,1},\ \ldots,\ \boldsymbol{k}_{N_t,s},\ \boldsymbol{\mu}) \leq 0 \\ & \boldsymbol{u}_0 - \bar{\boldsymbol{u}}(\boldsymbol{\mu}) = 0 \\ & \boldsymbol{u}_n - \boldsymbol{u}_{n-1} - \sum_{i=1}^{s} b_i \boldsymbol{k}_{n,i} = 0 \\ & \boldsymbol{M}\boldsymbol{k}_{n,i} - \Delta t_n(\boldsymbol{\mu}) \boldsymbol{r}\left(\boldsymbol{u}_{n,i},\ \boldsymbol{\mu},\ t_{n,i}(\boldsymbol{\mu})\right) = 0 \end{aligned}$$

Highlights of globally high-order discretization

- **Arbitrary Lagrangian-Eulerian** formulation:
  Map, $\mathcal{G}(\cdot, \boldsymbol{\mu}, t)$, from physical $v(\boldsymbol{\mu}, t)$ to reference V

$$\left.\frac{\partial \boldsymbol{U_X}}{\partial t}\right|_{\boldsymbol{X}} + \nabla_{\boldsymbol{X}} \cdot \boldsymbol{F_X}(\boldsymbol{U_X},\ \nabla_{\boldsymbol{X}} \boldsymbol{U_X}) = 0$$

- **Space discretization**: discontinuous Galerkin

$$\boldsymbol{M}\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{r}(\boldsymbol{u}, \boldsymbol{\mu}, t)$$

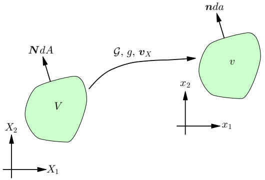
- **Time discretization**: diagonally implicit RK

$$\boldsymbol{u}_n = \boldsymbol{u}_{n-1} + \sum_{i=1}^{s} b_i \boldsymbol{k}_{n,i}$$

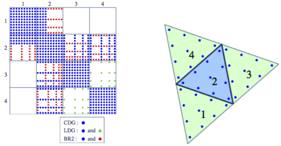
$$\boldsymbol{M}\boldsymbol{k}_{n,i} = \Delta t_n(\boldsymbol{\mu})\boldsymbol{r}\left(\boldsymbol{u}_{n,i},\ \boldsymbol{\mu},\ t_{n,i}(\boldsymbol{\mu})\right)$$

- **Quantity of interest**: solver-consistency

$$F(\boldsymbol{u}_0, \ldots, \boldsymbol{u}_{N_t}, \boldsymbol{k}_{1,1}, \ldots, \boldsymbol{k}_{N_t,s})$$



Mapping-Based ALE



DG Discretization

$$\begin{array}{c|ccccc}
c_1 & a_{11} & & & \\
c_2 & a_{21} & a_{22} & & \\
\vdots & \vdots & \vdots & \ddots & \\
c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\
\hline
 & b_1 & b_2 & \cdots & b_s
\end{array}$$

Butcher Tableau for DIRK

Adjoint method to efficiently compute gradients of QoI

- Consider the *fully discrete* output functional $F(\boldsymbol{u}_n, \boldsymbol{k}_{n,i}, \boldsymbol{\mu})$
    - Represents either the **objective** function or a **constraint**
- The *total derivative* with respect to the parameters $\boldsymbol{\mu}$, required in the context of gradient-based optimization, takes the form

$$\frac{\mathrm{d}F}{\mathrm{d}\boldsymbol{\mu}} = \frac{\partial F}{\partial \boldsymbol{\mu}} + \sum_{n=0}^{N_t} \frac{\partial F}{\partial \boldsymbol{u}_n} \frac{\partial \boldsymbol{u}_n}{\partial \boldsymbol{\mu}} + \sum_{n=1}^{N_t} \sum_{i=1}^{s} \frac{\partial F}{\partial \boldsymbol{k}_{n,i}} \frac{\partial \boldsymbol{k}_{n,i}}{\partial \boldsymbol{\mu}}$$

- The sensitivities, $\frac{\partial \boldsymbol{u}_n}{\partial \boldsymbol{\mu}}$ and $\frac{\partial \boldsymbol{k}_{n,i}}{\partial \boldsymbol{\mu}}$, are expensive to compute, requiring the solution of $n_{\boldsymbol{\mu}}$ linear evolution equations
- **Adjoint method**: alternative method for computing $\frac{\mathrm{d}F}{\mathrm{d}\boldsymbol{\mu}}$ that require one linear evolution evoluation equation for each quantity of interest, F

Adjoint equation derivation: outline

- Define **auxiliary** PDE-constrained optimization problem

$$
\begin{aligned}
\underset{\substack{\boldsymbol{u}_0,\ \ldots,\ \boldsymbol{u}_{N_t} \in \mathbb{R}^{N_{\boldsymbol{u}}},\\ \boldsymbol{k}_{1,1},\ \ldots,\ \boldsymbol{k}_{N_t,s} \in \mathbb{R}^{N_{\boldsymbol{u}}}}}{\text{minimize}} \quad & F(\boldsymbol{u}_0,\ \ldots,\ \boldsymbol{u}_{N_t},\ \boldsymbol{k}_{1,1},\ \ldots,\ \boldsymbol{k}_{N_t,s},\ \boldsymbol{\mu}) \\
\text{subject to} \quad & \boldsymbol{R}_0 = \boldsymbol{u}_0 - \bar{\boldsymbol{u}}(\boldsymbol{\mu}) = 0 \\
& \boldsymbol{R}_n = \boldsymbol{u}_n - \boldsymbol{u}_{n-1} - \sum_{i=1}^{s} b_i \boldsymbol{k}_{n,i} = 0 \\
& \boldsymbol{R}_{n,i} = \boldsymbol{M}\boldsymbol{k}_{n,i} - \Delta t_n \boldsymbol{r}\left(\boldsymbol{u}_{n,i},\ \boldsymbol{\mu},\ t_{n,i}\right) = 0
\end{aligned}
$$

- Define **Lagrangian**

$$
\mathcal{L}(\boldsymbol{u}_n,\ \boldsymbol{k}_{n,i},\ \boldsymbol{\lambda}_n,\ \boldsymbol{\kappa}_{n,i}) = F - \boldsymbol{\lambda}_0{}^T \boldsymbol{R}_0 - \sum_{n=1}^{N_t} \boldsymbol{\lambda}_n{}^T \boldsymbol{R}_n - \sum_{n=1}^{N_t} \sum_{i=1}^{s} \boldsymbol{\kappa}_{n,i}{}^T \boldsymbol{R}_{n,i}
$$

- The solution of the optimization problem is given by the **Karush-Kuhn-Tucker (KKT) sytem**

$$
\frac{\partial \mathcal{L}}{\partial \boldsymbol{u}_n} = 0, \quad \frac{\partial \mathcal{L}}{\partial \boldsymbol{k}_{n,i}} = 0, \quad \frac{\partial \mathcal{L}}{\partial \boldsymbol{\lambda}_n} = 0, \quad \frac{\partial \mathcal{L}}{\partial \boldsymbol{\kappa}_{n,i}} = 0
$$

Dissection of fully discrete adjoint equations

- **Linear** evolution equations solved **backward** in time
- **Primal** state/stage, $\boldsymbol{u}_{n,i}$ required at each state/stage of dual problem
- Heavily dependent on **chosen ouput**

$$\boldsymbol{\lambda}_{N_t} = \frac{\partial F}{\partial \boldsymbol{u}_{N_t}}^T$$

$$\boldsymbol{\lambda}_{n-1} = \boldsymbol{\lambda}_n + \frac{\partial F}{\partial \boldsymbol{u}_{n-1}}^T + \sum_{i=1}^{s} \Delta t_n \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{u}} (\boldsymbol{u}_{n,i},\ \boldsymbol{\mu},\ t_{n-1} + c_i \Delta t_n)^T \boldsymbol{\kappa}_{n,i}$$

$$\boldsymbol{M}^T \boldsymbol{\kappa}_{n,i} = \frac{\partial F}{\partial \boldsymbol{k}_{n,i}}^T + b_i \boldsymbol{\lambda}_n + \sum_{j=i}^{s} a_{ji} \Delta t_n \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{u}} (\boldsymbol{u}_{n,j},\ \boldsymbol{\mu},\ t_{n-1} + c_j \Delta t_n)^T \boldsymbol{\kappa}_{n,j}$$

- Gradient reconstruction via dual variables

$$\frac{\mathrm{d}F}{\mathrm{d}\boldsymbol{\mu}} = \frac{\partial F}{\partial \boldsymbol{\mu}} + \boldsymbol{\lambda}_0{}^T \frac{\partial \bar{\boldsymbol{u}}}{\partial \boldsymbol{\mu}}(\boldsymbol{\mu}) + \sum_{n=1}^{N_t} \Delta t_n \sum_{i=1}^{s} \boldsymbol{\kappa}_{n,i}{}^T \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{\mu}}(\boldsymbol{u}_{n,i},\ \boldsymbol{\mu},\ t_{n,i})$$

Dissection of fully discrete adjoint equations

Parametrized time domain: *modifies gradient reconstruction from adjoint solution, not adjoint equations themselves*

$$\begin{aligned}\frac{\mathrm{d}F}{\mathrm{d}\boldsymbol{\mu}} &= \frac{\partial F}{\partial \boldsymbol{\mu}} + \boldsymbol{\lambda}_0^T \frac{\partial \bar{\boldsymbol{u}}}{\partial \boldsymbol{\mu}}(\boldsymbol{\mu}) + \sum_{n=1}^{N_t} \Delta t_n \sum_{i=1}^{s} \boldsymbol{\kappa}_{n,i}^T \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{\mu}}(\boldsymbol{u}_{n,i},\ \boldsymbol{\mu},\ t_{n,i}) \\ &+ \sum_{n=1}^{N_t} \sum_{i=1}^{s} b_i \left[\Delta t_n \frac{\partial f^h}{\partial t}(\boldsymbol{u}_{n,i},\ \boldsymbol{\mu},\ t_{n,i}) \frac{\partial t_{n,i}}{\partial \boldsymbol{\mu}}(\boldsymbol{\mu}) + f^h(\boldsymbol{u}_{n,i},\ \boldsymbol{\mu},\ t_{n,i}) \frac{\partial \Delta t_n}{\partial \boldsymbol{\mu}}(\boldsymbol{\mu}) \right] \\ &+ \sum_{n=1}^{N_t} \sum_{i=1}^{s} \boldsymbol{\kappa}_{n,i}^T \left[\Delta t_n \frac{\partial \boldsymbol{r}}{\partial t}(\boldsymbol{u}_{n,i},\ \boldsymbol{\mu},\ t_{n,i}) \frac{\partial t_{n,i}}{\partial \boldsymbol{\mu}}(\boldsymbol{\mu}) + \boldsymbol{r}(\boldsymbol{u}_{n,i},\ \boldsymbol{\mu},\ t_{n,i}) \frac{\partial \Delta t_n}{\partial \boldsymbol{\mu}}(\boldsymbol{\mu}) \right]\end{aligned}$$

where $f^h(\boldsymbol{u},\ \boldsymbol{\mu},\ t)$ is DG approximation to $\int_{\boldsymbol{\Gamma}} j(\boldsymbol{U},\ \boldsymbol{\mu},\ t)\, dS$ and

$$F(\boldsymbol{u}_0,\ \ldots,\ \boldsymbol{u}_{N_t},\ \boldsymbol{k}_{1,1},\ \ldots,\ \boldsymbol{k}_{N_t,s},\ \boldsymbol{\mu}) = \sum_{n=1}^{N_t} \Delta t_n(\boldsymbol{\mu}) \sum_{i=1}^{s} b_i f^h(\boldsymbol{u}_{n,i},\ \boldsymbol{\mu},\ t_{n,i}(\boldsymbol{\mu}))$$

Implementation details

- Implementation of the fully discrete adjoint method relies on the computation of the following terms from the **spatial discretization**

$$\boldsymbol{M}, \boldsymbol{r}, \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{u}}, \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{\mu}}, \frac{\partial \boldsymbol{r}}{\partial t}, f^h, \frac{\partial f^h}{\partial \boldsymbol{u}}, \frac{\partial f^h}{\partial \boldsymbol{\mu}}, \frac{\partial f^h}{\partial t},$$

  and terms from the **temporal discretization**

$$t_{n,i}, \Delta t_n, \frac{\partial t_{n,i}}{\partial \boldsymbol{\mu}}, \frac{\partial \Delta t_n}{\partial \boldsymbol{\mu}}.$$

- In the case of deforming domain problems treated with **ALE** formulation:

$$\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{u}, \boldsymbol{x}(\boldsymbol{\mu}, t), \dot{\boldsymbol{x}}(\boldsymbol{\mu}, t))$$
$$f^h = f^h(\boldsymbol{u}, \boldsymbol{x}(\boldsymbol{\mu}, t), \dot{\boldsymbol{x}}(\boldsymbol{\mu}, t))$$

- Partial derivatives w.r.t. $\boldsymbol{\mu}$ and t computed as:

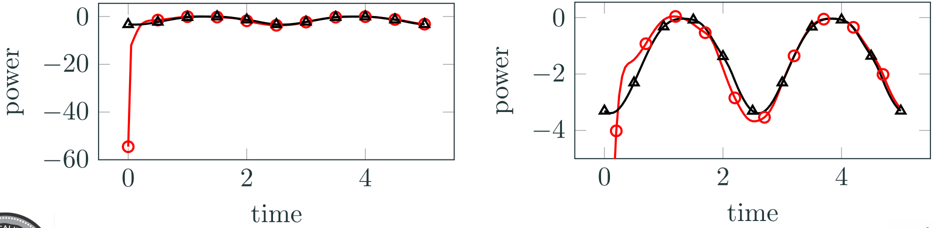
$$\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{\mu}} = \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\mu}} + \frac{\partial \boldsymbol{r}}{\partial \dot{\boldsymbol{x}}}\frac{\partial \dot{\boldsymbol{x}}}{\partial \boldsymbol{\mu}} \qquad \frac{\partial f_h}{\partial \boldsymbol{\mu}} = \frac{\partial f_h}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\mu}} + \frac{\partial f_h}{\partial \dot{\boldsymbol{x}}}\frac{\partial \dot{\boldsymbol{x}}}{\partial \boldsymbol{\mu}}$$
$$\frac{\partial \boldsymbol{r}}{\partial t} = \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{x}}{\partial t} + \frac{\partial \boldsymbol{r}}{\partial \dot{\boldsymbol{x}}}\frac{\partial \dot{\boldsymbol{x}}}{\partial t} \qquad \frac{\partial f_h}{\partial t} = \frac{\partial f_h}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{x}}{\partial t} + \frac{\partial f_h}{\partial \dot{\boldsymbol{x}}}\frac{\partial \dot{\boldsymbol{x}}}{\partial t}$$

Time-periodic solutions desired when optimizing cyclic motion

- To properly optimize a cyclic, or periodic problem, need to simulate a **representative** period
- Necessary to avoid transients that will impact quantity of interest and may cause simulation to crash
- **Task**: Find initial condition, $\bar{\boldsymbol{u}}$, such that flow is periodic, i.e. $\boldsymbol{u}_{N_t} = \bar{\boldsymbol{u}}$

Time-periodic solutions desired when optimizing cyclic motion

Vorticity around airfoil with flow initialized from steady-state (left) and time-periodic flow (right)



Time history of power on airfoil of flow initialized from steady-state ( ) and from a time-periodic solution ( )

Recall *fully discrete* PDE-constrained optimization problem

$$
\begin{array}{ll}
\underset{\substack{\boldsymbol{u}_0,\ \ldots,\ \boldsymbol{u}_{N_t} \in \mathbb{R}^{N_{\boldsymbol{u}}},\\ \boldsymbol{k}_{1,1},\ \ldots,\ \boldsymbol{k}_{N_t,s} \in \mathbb{R}^{N_{\boldsymbol{u}}},\\ \boldsymbol{\mu} \in \mathbb{R}^{n_{\boldsymbol{\mu}}}}}{\text{minimize}} & J(\boldsymbol{u}_0,\ \ldots,\ \boldsymbol{u}_{N_t},\ \boldsymbol{k}_{1,1},\ \ldots,\ \boldsymbol{k}_{N_t,s},\ \boldsymbol{\mu}) \\
\text{subject to} & \mathbf{C}(\boldsymbol{u}_0,\ \ldots,\ \boldsymbol{u}_{N_t},\ \boldsymbol{k}_{1,1},\ \ldots,\ \boldsymbol{k}_{N_t,s},\ \boldsymbol{\mu}) \leq 0 \\
& \boldsymbol{u}_0 - \bar{\boldsymbol{u}}(\boldsymbol{\mu}) = 0 \\
& \boldsymbol{u}_n - \boldsymbol{u}_{n-1} + \displaystyle\sum_{i=1}^{s} b_i \boldsymbol{k}_{n,i} = 0 \\
& \boldsymbol{M}\boldsymbol{k}_{n,i} - \Delta t_n \boldsymbol{r}\left(\boldsymbol{u}_{n,i},\ \boldsymbol{\mu},\ t_{n,i}\right) = 0
\end{array}
$$

Time-periodicity constraints in PDE-constrained optimization

Slight modification leads to fully discrete periodic PDE-constrained optimization

$$
\begin{aligned}
\underset{\substack{\boldsymbol{u}_0,\,\ldots,\,\boldsymbol{u}_{N_t}\in\mathbb{R}^{N_{\boldsymbol{u}}},\\ \boldsymbol{k}_{1,1},\,\ldots,\,\boldsymbol{k}_{N_t,s}\in\mathbb{R}^{N_{\boldsymbol{u}}},\\ \boldsymbol{\mu}\in\mathbb{R}^{n_{\boldsymbol{\mu}}}}}{\text{minimize}} \quad & J(\boldsymbol{u}_0,\ \ldots,\ \boldsymbol{u}_{N_t},\ \boldsymbol{k}_{1,1},\ \ldots,\ \boldsymbol{k}_{N_t,s},\ \boldsymbol{\mu}) \\
\text{subject to} \quad & \mathbf{C}(\boldsymbol{u}_0,\ \ldots,\ \boldsymbol{u}_{N_t},\ \boldsymbol{k}_{1,1},\ \ldots,\ \boldsymbol{k}_{N_t,s},\ \boldsymbol{\mu}) \leq 0 \\
& \boldsymbol{u}_0 - \boldsymbol{u}_{N_t} = 0 \\
& \boldsymbol{u}_n - \boldsymbol{u}_{n-1} + \sum_{i=1}^{s} b_i \boldsymbol{k}_{n,i} = 0 \\
& \boldsymbol{M}\boldsymbol{k}_{n,i} - \Delta t_n \boldsymbol{r}\left(\boldsymbol{u}_{n,i},\ \boldsymbol{\mu},\ t_{n,i}\right) = 0
\end{aligned}
$$

Adjoint method for periodic PDE-constrained optimization

- Following identical procedure as for non-periodic case, the adjoint equations corresponding to the periodic conservation law are

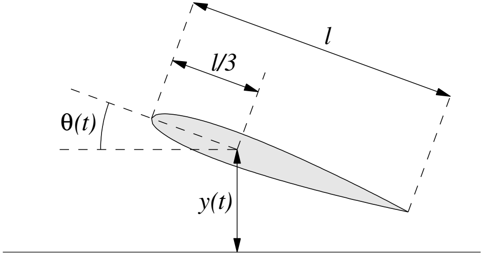
$$
\begin{aligned}
\boldsymbol{\lambda}_{N_t} &= {\color{red}\boldsymbol{\lambda}_0} + \frac{\partial F}{\partial \boldsymbol{u}_{N_t}}^T \\
\boldsymbol{\lambda}_{n-1} &= \boldsymbol{\lambda}_n + \frac{\partial F}{\partial \boldsymbol{u}_{n-1}}^T + \sum_{i=1}^{s} \Delta t_n \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{u}} \left(\boldsymbol{u}_{n,i},\ \boldsymbol{\mu},\ t_{n-1} + c_i \Delta t_n\right)^T \boldsymbol{\kappa}_{n,i} \\
\boldsymbol{M}^T \boldsymbol{\kappa}_{n,i} &= \frac{\partial F}{\partial \boldsymbol{u}_{N_t}}^T + b_i \boldsymbol{\lambda}_n + \sum_{j=i}^{s} a_{ji} \Delta t_n \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{u}} \left(\boldsymbol{u}_{n,j},\ \boldsymbol{\mu},\ t_{n-1} + c_j \Delta t_n\right)^T \boldsymbol{\kappa}_{n,j}
\end{aligned}
$$

- Dual problem is also periodic
- Solve *linear, periodic* problem using Krylov shooting method

Energetically optimal flapping under thrust constraint

$$\begin{aligned}
\underset{\boldsymbol{\mu}}{\text{minimize}} \quad & -\frac{1}{T(\boldsymbol{\mu})}\int_0^{T(\boldsymbol{\mu})}\int_{\boldsymbol{\Gamma}} \boldsymbol{f}\cdot\boldsymbol{v}\,dS\,dt \\
\text{subject to} \quad & \frac{1}{T(\boldsymbol{\mu})}\int_0^{T(\boldsymbol{\mu})}\int_{\boldsymbol{\Gamma}} \boldsymbol{f}\cdot\boldsymbol{e}_1\,dS\,dt = q \\
& \frac{\partial \boldsymbol{U}}{\partial t} + \nabla\cdot\boldsymbol{F}(\boldsymbol{U},\nabla\boldsymbol{U}) = 0
\end{aligned}$$

- Isentropic, compressible, Navier-Stokes
- Re = 1000, M = 0.02
- $y(t)$, $\theta(t)$ parametrized by single harmonic term
- Black-box optimizer: IPOPT



Airfoil schematic, kinematic description

Energetically optimal flapping vs. required thrust

Initial Guess	Optimal $T_x = 0$	Optimal $T_x = 1.0$
$\text{Energy} = 1.8445$	$\text{Energy} = 0.21934$	$\text{Energy} = 1.93826$
$\text{Thrust} = 0.06729$	$\text{Thrust} = 0.0000$	$\text{Thrust} = 1.0000$

Energetically optimal flapping vs. required thrust

Initial Guess	Optimal $T_x = 0$	Optimal $T_x = 1.5$
Energy $= 1.8445$	Energy $= 0.21934$	Energy $= 3.00404$
Thrust $= 0.06729$	Thrust $= 0.0000$	Thrust $= 1.5000$

Energetically optimal flapping vs. required thrust

Initial Guess	Optimal $T_x = 0$	Optimal $T_x = 2.0$
$\text{Energy} = 1.8445$	$\text{Energy} = 0.21934$	$\text{Energy} = 4.6522$
$\text{Thrust} = 0.06729$	$\text{Thrust} = 0.0000$	$\text{Thrust} = 2.0000$

Energetically optimal flapping vs. required thrust

$$\text{Energy} = 1.8445 \quad \text{Thrust} = 0.06729$$

$$\text{Energy} = 0.21934 \quad \text{Thrust} = 0.0000$$

$$\text{Energy} = 6.2869 \quad \text{Thrust} = 2.5000$$

Initial Guess

Optimal $T_x = 0$

Optimal $T_x = 2.5$

Energetically optimal flapping vs. required thrust

$$\text{Energy} = 0.21935$$
$$\text{Thrust} = 0.0000$$

$$\text{Energy} = 3.00404$$
$$\text{Thrust} = 1.5000$$

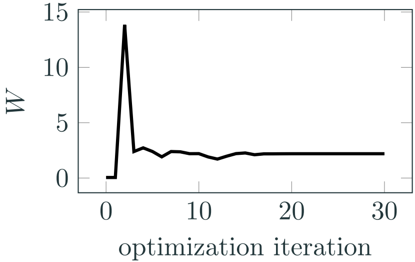
$$\text{Energy} = 6.2869$$
$$\text{Thrust} = 2.5000$$

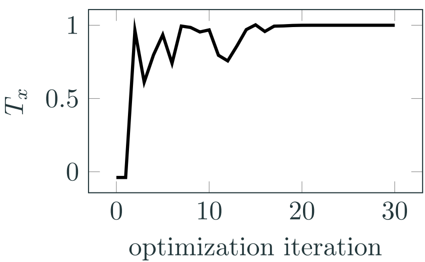
Optimal $T_x = 0$

Optimal
$T_x = 1.5$

Optimal
$T_x = 2.5$

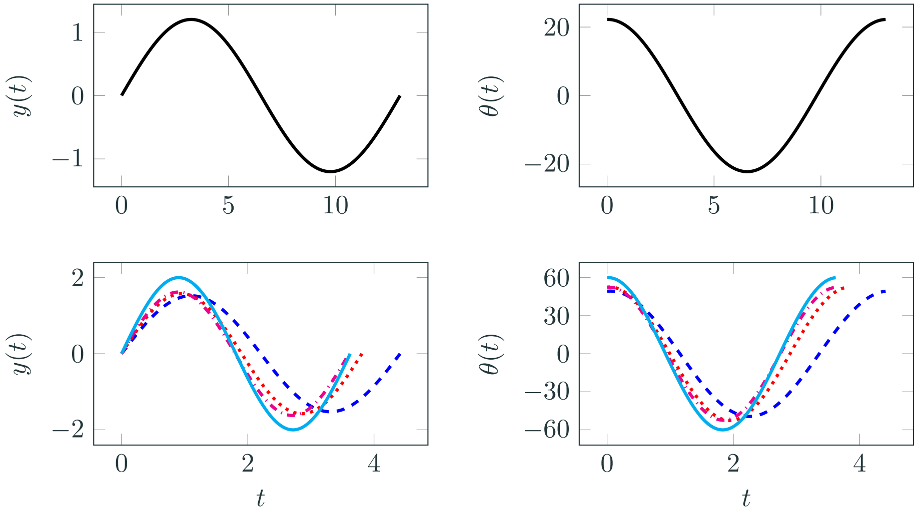
Rapid convergence of optimizer to feasible minimizer





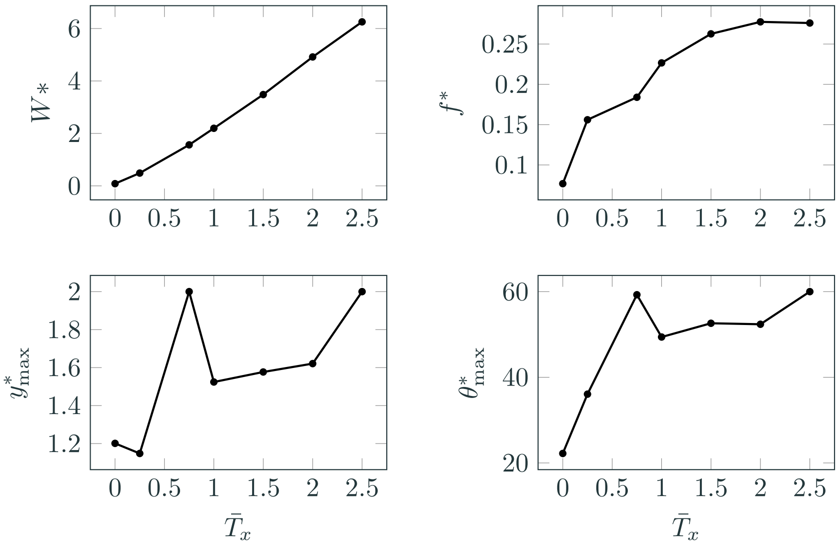
Convergence of required flapping energy and thrust as a function of optimization iteration corresponding to the thrust constraint $\bar{T}_x = 1$. The final values are $W^* = 2.1961022$ and $T_x^* = 0.9999999$ and the first-order optimality conditions are satisfied to a tolerance of 10^{-8}. For a convergence tolerance of 10^{-4}, the optimization iterations could have been terminated after 20 iterations.

Energetically optimal flapping vs. required thrust: Trajectory



Optimal trajectories of $y(t)$ and $\theta(t)$ for various value of the thrust constraint: $\bar{T}_x = 0.0$ (solid black), $\bar{T}_x = 1.0$ (dashed blue), $\bar{T}_x = 1.5$ (dotted red), $\bar{T}_x = 2.0$ (dash-dot magenta), $\bar{T}_x = 2.5$ (solid cyan).

Energetically optimal flapping vs. required thrust: QoI



The optimal flapping energy (W^*), frequency (f^*), maximum heaving amplitude (y^*_{max}), and maximum pitching amplitude (θ^*_{max}) as a function of the thrust constraint $\bar{T}_x$.

Summary and future work

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- Extended standard fully discrete adjoint framework to handle parametrization that affects the **time discretization**
- Alters **reconstruction** of $\nabla_{\boldsymbol{\mu}} F$ from adjoint solution, **not adjoint equations**
- Implementation requires **velocity** of quantity of interest and residual
- Framework used to study energetically optimal flight
  - Optimal energy approximately linear in required thrust
  - Optimal frequency increases then plateaus as a function of the required thrust

Future work

- Study 3D flapping with frequency optimization
- Extension to general time domain/discretization parametrization
  - Trajectory/path optimization